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Department of General Physics and Solid State Physics

Chursanova M.V.

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CHAPTER 1. PHYSICAL PRINCIPLES OF MECHANICS

Topic 1.1. Kinematics of translational and rotational motion

Lecture 1

1.1.1 Classical mechanics

Classical mechanics (or Newtonian mechanics) is the branch of physics which describes the motion of bodies when their velocities are much smaller than the speed of light. In this case linear dimensions and time intervals do not depend on the choice of the reference frame, and the Newtonian conception of the absolute time and space is applicable.

Motion of an object is a change in the object's position in the three-dimensional space. Any kind of motion can be described as a combination of two distinct types of motion: translational motion and rotational motion:

1. **Translational motion** occurs when the path traced out by any particle is exactly parallel to the path traced out by every other particle in the body;
2. **Rotational motion** occurs if every particle in the body moves in a circle about a single line. This line is called the axis of rotation.

In order to describe the motion the **reference frame** is necessary. Reference frame consists of a coordinate system, a time measurer and the set of physical reference points that uniquely fix (locate and orient) the coordinate system.

The **coordinate system** is used to describe the position of a point in space. It consists of:

1. An origin at a particular point in space

2. A set of coordinate axes with scales and labels

3. Choice of positive direction for each axis: unit vectors

The **Cartesian coordinate system** is commonly chosen. In a three-dimensional space it consists of triplet axes that are pair-wise perpendicular, have a single unit of length for all three axes and have an orientation for each axis.

1.1.2 Kinematics

Kinematics is the branch of classical mechanics which describes the motion of bodies without consideration of the reasons that cause this motion. We describe the motion of an object while ignoring the interactions with external agents that might be causing or modifying that motion.

– **Material point** (alternatively, a classical **particle**) is the body, whose size may be neglected in the given conditions.
– **Rigid body** is the body in which all the distances between the component particles are constant. In other words, a rigid body is a solid whose deformation may be neglected in the given conditions.

The three basic kinematic variables are:

1. the position of an object: its location in space
2. the velocity of an object: how fast it is changing its position
3. the acceleration of an object: how fast the velocity is changing

1.1.2.1 Position

Position vector (or **radius vector**) $\vec{r}$ is a vector showing the position of the object. It is drawn from the origin of the reference frame to that object (Figure 1.1). **Displacement** $\Delta\vec{r}$ is a vector connecting the initial and the final position of the object. **Trajectory** is the locus of the ends of the radius vector during the object's motion. It is the line that a moving object follows through space. **Path** is the length of the trajectory line.

Time dependence of the radius vector $\vec{r} = \vec{r}(t)$ is the **law of motion** of the object, which allows to determine its position, velocity and acceleration at any moment of time.

Let the $x = x(t)$, $y = y(t)$ and $z = z(t)$ be coordinates of the object in the Cartesian coordinate system. Then

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \qquad (1.1)$$

where $\vec{i}, \vec{j}, \vec{k}$ are the pair-wise perpendicular unit vectors, $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$.

The magnitude of displacement

$$|\Delta\vec{r}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, \qquad (1.2)$$

where x_1, y_1, z_1 and x_2, y_2, z_2 are coordinates of initial and final positions of the object.

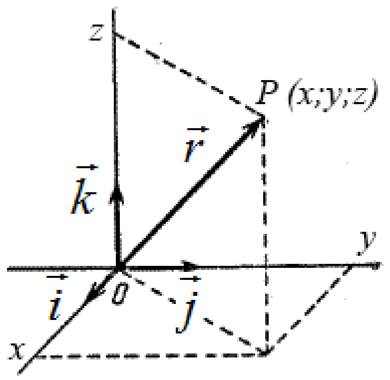


Figure 1.1 Position vector of the point P in the Cartesian coordinate system.

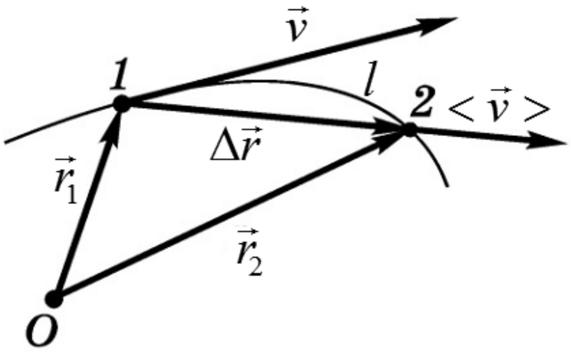


Figure 1.2 Displacement $\Delta\vec{r}$, path l, average and instanteneous velocity of the particle.

1.1.2.2 Velocity

Velocity $\vec{v}$ is the rate of change of the particle's position vector with respect to time.

The **average velocity** is defined as

$\langle\vec{v}\rangle = \dfrac{\Delta\vec{r}}{\Delta t}$, where $\Delta\vec{r}$ is the change in the position vector over the time interval Δt.

The direction of the average velocity vector is the same as the direction of the displacement vector (see Figure 1.2).

The **instantaneous velocity**:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}. \tag{1.3}$$

The velocity of the object at the given instant of time is defined as the time derivative of the radius vector.

The direction of the velocity vector is tangent to the trajectory of the particle at every position the particle occupies along its path (see Figure 1.2).

The coordinates of the velocity vector $v_x = \frac{dx}{dt},\ v_y = \frac{dy}{dt},\ v_z = \frac{dz}{dt}$, and

$$\vec{v}(t) = v_x(t)\vec{i} + v_y(t)\vec{j} + v_z(t)\vec{k}. \tag{1.4}$$

The magnitude of the velocity is called the **speed** of the object:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}. \tag{1.5}$$

Speed is the distance traveled by the particle in unit time. It is a scalar quantity. Notice the clear distinction between the definitions of average velocity and average speed: average velocity is the displacement divided by the time interval, whereas average speed is the distance divided by the time interval.

Rectilinear uniform motion: the object travels in a straight line and covers equal distances in equal intervals of time. The velocity is constant $\vec{v} = const$. If an object travels a distance l in time t, then its speed is $v = \frac{l}{t}$ in the case of rectilinear uniform motion.

1.1.2.3 Acceleration

Acceleration is the rate of change in the velocity of the object per unit time

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}. \tag{1.6}$$

The acceleration of the object at the given instant of time is defined as the time derivative of its velocity. The acceleration is positive when it is in the direction of velocity, and negative when it is opposite to the direction of velocity.

The coordinates of the acceleration vector

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2},\ a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2},\ a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}, \text{ and}$$

$$\vec{a}(t) = a_x(t)\vec{i} + a_y(t)\vec{j} + a_z(t)\vec{k}\,. \qquad (1.7)$$

The magnitude of the acceleration:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}\,. \qquad (1.8)$$

<u>Rectilinear uniformly accelerated motion</u>: the object travels in a straight line and its velocity increases or decreases by equal amounts in equal intervals of time. The acceleration is constant $\vec{a} = const$.

Now let's consider an inverse problem of finding velocity and position vector of the object if its acceleration is given. The initial conditions, such as the velocity $\vec{v}_0$ and position at the initial moment of time, are necessary for that.

Let's find the velocity:

$$d\vec{v} = \vec{a}dt; \qquad \int_{v_0}^{v} d\vec{v} = \int_0^t \vec{a}dt; \qquad \vec{v} - \vec{v}_0 = \int_0^t \vec{a}dt\,.$$

In the case of uniformly accelerated motion $\vec{a} = const$, and

$$\vec{v} = \vec{v}_0 + \vec{a}t\,. \qquad (1.9)$$

Let's find the position:

$$d\vec{r} = \vec{v}dt; \qquad \int_{r_0}^{r} d\vec{r} = \int_0^t \vec{v}dt; \qquad \vec{r} - \vec{r}_0 = \int_0^t \vec{v}dt\,.$$

In the case of uniformly accelerated motion, $\vec{r} - \vec{r}_0 = \int_0^t (\vec{v}_0 + \vec{a}t)dt$ and

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{\vec{a}t^2}{2}. \qquad (1.10)$$

1.1.3 Galilean law of velocity addition

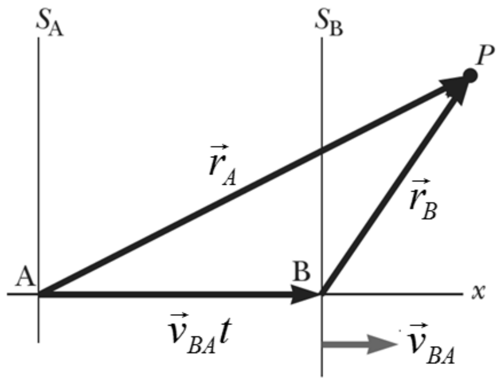


Figure 1.3.

Let's consider a situation when motion of the particle is being described by two observers in two different frames of reference, observer A is in a reference frame S_A fixed relative to the Earth and a second observer B in a reference frame S_B moving to the right relative to S_A (and therefore relative to the Earth) with a constant velocity $\vec{v}_{BA}$ (Figure 1.3).

We define the time $t = 0$ as the instant at which the origins of the two reference frames coincide in space. Therefore, at time t, the origins of the reference frames will be separated by a distance $v_{BA} \cdot t$. Then the position vectors of the particle relative to these two reference frames will be

$$\vec{r}_A = \vec{r}_B + \vec{v}_{BA} t. \qquad (1.11)$$

By differentiating this equation with respect to time, we obtain the **Galilean velocity transformation law**:

$$\vec{v}_A = \vec{v}_B + \vec{v}_{BA}, \qquad (1.12)$$

where $\vec{v}_A$ is the velocity of the object relative to the frame S_A, $\vec{v}_B$ is the velocity of the object relative to the frame S_B, $\vec{v}_{BA}$ is velocity of the S_B frame relative to the S_A frame.

Although observers in two frames measure different velocities for the particle, they measure the same acceleration as far as $\vec{v}_{BA} = const$ and $\dfrac{d\vec{v}_{BA}}{dt} = 0$.

1.1.4 Projectile Motion

Projectile motion is a form of motion in which a body or particle (called a projectile) is thrown near the earth's surface, and it moves under the action of gravity only.

So, the projectile acceleration is that due to gravity,

$\vec{a} = \vec{g}$, $g = 9.8$ m/s^2

There are two assumptions:

1) The free-fall acceleration is constant over the range of motion and is directed downward;
2) The air resistance is negligible.

This is the case of **two-dimensional motion**, which can be analyzed as a combination of two independent motions in the x and y directions (see Figure 1.4).

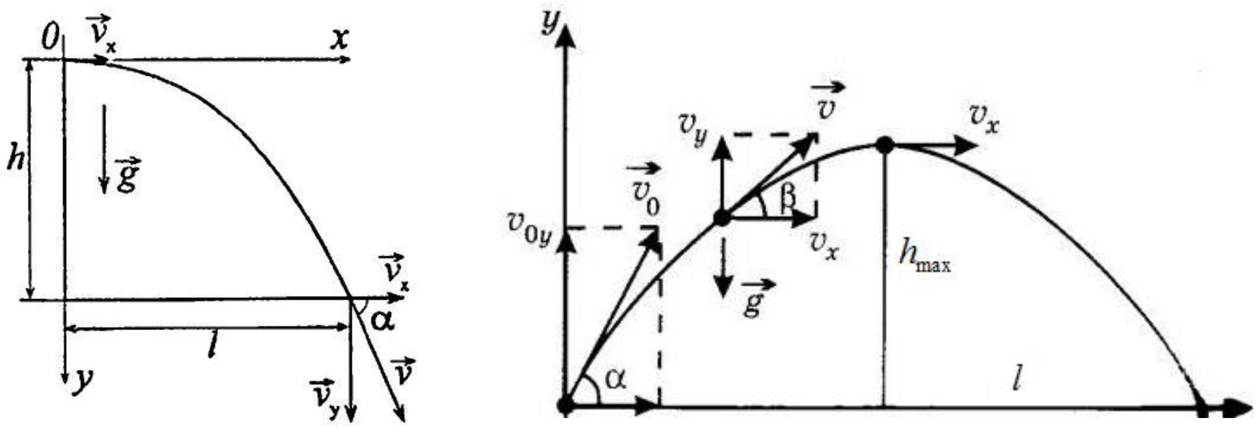


Figure 1.4: (a) Motion of the projectile thrown horizontally from a certain height *h*; (b) Motion of the projectile thrown from the origin at an angle *α* above the horizontal

Horizontal and vertical **components of the acceleration**:

$a_x = 0$; $a_y = -g$

So, the projectile motion is superposition of two motions: (1) uniform motion of a particle under constant velocity in the horizontal direction and (2) uniformly accelerated motion of a particle under constant acceleration (free fall) in the vertical direction.

Horizontal and vertical **components of the velocity**:

$$v_x = v_{0x} + a_x t\,; \qquad v_y = v_{0y} + a_y t$$

Motion of the projectile thrown horizontally from a certain height	*Motion of the projectile thrown from the origin at an angle above the horizontal*
$v_{0x} = v_0\,; \quad v_{0y} = 0\,;$ So, $v_x = v_0\,;\ v_y = -gt$ The **magnitude of the velocity**: $v = \sqrt{v_o^2 + g^2 t^2}$	$v_{0x} = v_0 \cos\alpha\,; \quad v_{0y} = v_0 \sin\alpha$ So, $v_x = v_0 \cos\alpha\,; \quad v_y = v_0 \sin\alpha - gt$ The **magnitude of the velocity**: $v = \sqrt{v_o^2 \cos^2\alpha + (v_0 \sin\alpha - gt)^2}$

Horizontal and vertical **coordinates**:

$$x = x_0 + v_{0x} t + \frac{a_x t^2}{2}\,; \qquad y = y_0 + v_{0y} t + \frac{a_y t^2}{2}$$

Motion of the projectile thrown horizontally from a certain height	*Motion of the projectile thrown from the origin at an angle above the horizontal*
$x = v_0 t\,; \quad y = y_0 - \frac{gt^2}{2}$ **Equation of the trajectory**: $y = y_0 - \frac{gx^2}{2v_0^2}.$ **Horizontal range**: $l = v_0 t\,.$	$x = v_0 \cos\alpha \cdot t\,; \quad y = v_0 \sin\alpha \cdot t - \frac{gt^2}{2}$ **Equation of the trajectory**: $y = tg\alpha \cdot x - \frac{gx^2}{2v_0^2 \cos^2\alpha}$

Time of flight: $t=\sqrt{\dfrac{2y_0}{g}}$	**Maximum height**: $h_{\max}=\dfrac{v_0^2\sin^2\alpha}{2g}$ **Horizontal range**: $l=\dfrac{v_0^2\sin 2\alpha}{g}$

1.1.5 Curvilinear motion

Let us consider a more general motion when a particle moves along a curved path, and its velocity changes both in direction and in magnitude. In this situation, the velocity vector is always tangent to the path:

$$\vec{v}=v_\tau\vec{\tau}\,, \tag{1.13}$$

where $\vec{\tau}$ is a unit vector whose direction is tangent to the trajectory *l*; $v_\tau=\dfrac{dl}{dt}$ is the speed of the particle (see Figure 1.5).

Then, the total acceleration of the particle: $\vec{a}=\dfrac{d\vec{v}}{dt}=\dfrac{dv_\tau}{dt}\vec{\tau}+v_\tau\dfrac{d\vec{\tau}}{dt}$.

The first term, $\vec{a}_\tau=\dfrac{dv_\tau}{dt}\vec{\tau}$ is the **tangential acceleration** due to the changing magnitude of the velocity (changing speed). This component is parallel to the instantaneous velocity.

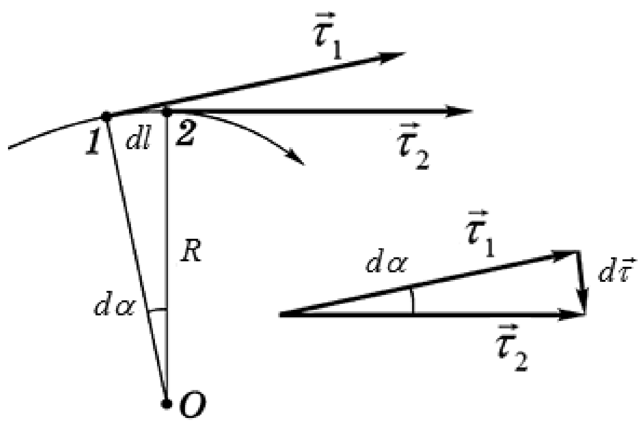


Figure 1.5.

The second term is

$$v_\tau\frac{d\vec{\tau}}{dt}=v_\tau\frac{d\vec{\tau}}{dl}\frac{dl}{dt}=v_\tau^2\frac{d\vec{\tau}}{dl}.$$

Now imagine that points 1 and 2 in Figure 1.5 become extremely close together. The radii-vectors at these points then can be considered of equal magnitude *R* called the radius of curvature of the trajectory. Then the circular path $dl = R\cdot d\alpha$, $|d\vec{\tau}| \approx |\vec{\tau}| d\alpha = 1 \cdot d\alpha = d\alpha$, and $\left|\frac{d\vec{\tau}}{dl}\right| = \frac{1}{R}$. Moreover, when dl approaches zero, $d\vec{\tau}$ is perpendicular to $\vec{\tau}$, that is $\frac{d\vec{\tau}}{dl}$ is directed along the normal to the trajectory at point 1. Let $\vec{n}$ be the unit vector which is normal to the trajectory at point 1 and directed to the center of the curvature.

So, we obtain $\vec{a}_r = \frac{v_\tau^2}{R}\vec{n}$, the **radial acceleration** due to the change in direction of the velocity. This component is directed perpendicular to the instantaneous velocity.

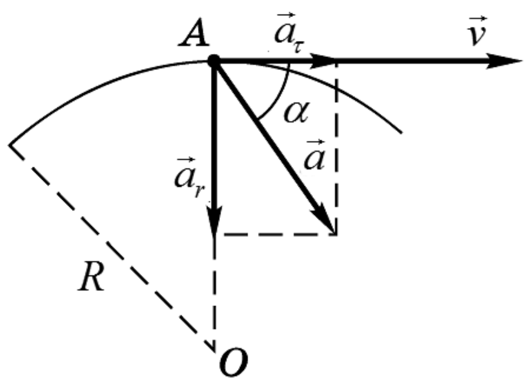


Figure 1.6.

The **total acceleration** is the vector sum of the tangential and radial components (Figure 1.6):

$$\vec{a} = \vec{a}_\tau + \vec{a}_r = \frac{dv_\tau}{dt}\vec{\tau} + \frac{v_\tau^2}{R}\vec{n}; \qquad (1.14)$$

$$|\vec{a}| = \sqrt{a_\tau^2 + a_r^2}. \qquad (1.15)$$

Circular motion is a particular case of the curvilinear motion when the particle moves along a circular path.

Uniform circular motion describes the motion of a body traveling a circular path at constant speed: $|\vec{v}_1| = |\vec{v}_2| = v$.

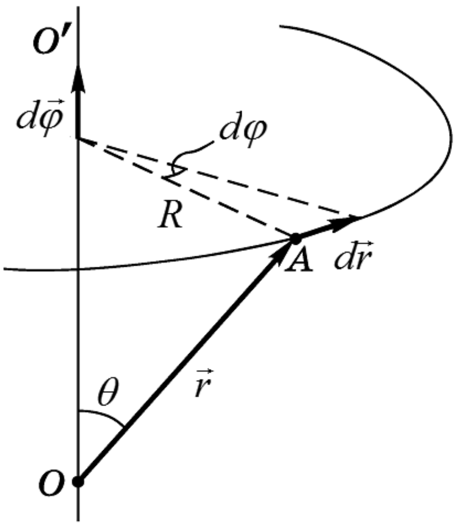
Uniform circular motion is the case when the body moves at a constant speed in a circular path, but still has an acceleration:

$\vec{a}_\tau = 0,\ \vec{a}_r \neq 0$.

In the case of the circular motion, the acceleration due to change in the direction is called **centripetal**. It is directed at all times towards the axis of rotation, and its magnitude is:

$$a_c = |a_r| = \frac{v^2}{R}. \qquad (1.16)$$

1.1.6 Kinematics of rotational motion



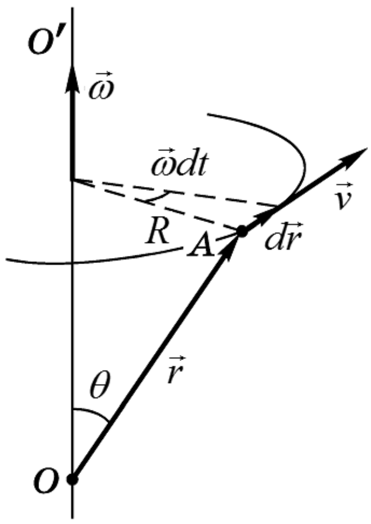
Let’s consider rotation of a body around a fixed axis. It is convenient to represent the position of the body with its polar coordinates (r, φ), where r is the distance from the origin to the body, and the angle φ is measured counterclockwise (Figure 1.7). In this representation, the angle φ changes in time while r remains constant.

Figure 1.7.

The angle φ is called the **angular position** of the body, while its change $d\varphi$ is the **angular displacement**. The angular displacement is considered as vector directed along the axis of rotation (see the Figure 1.7).

Then the linear displacement $|dr| = r\cdot\sin\theta\cdot d\varphi$, or in the vector form, it is the vector product of the angular displacement vector and the radius vector:

$$d\vec{r} = \left[d\vec{\varphi}, \vec{r}\right] \tag{1.17}$$



Angular velocity, also known as angular frequency, is the angular displacement in unit time (Figure 1.8):

$$\vec{\omega} = \frac{d\vec{\varphi}}{dt}. \tag{1.18}$$

Then the linear velocity $\vec{v} = \left[\vec{\omega}, \vec{r}\right]$. Its magnitude $v = \omega r\sin\theta$, or $v = \omega R$, where R is radius of the circle traced out by the body.

Figure 1.8.

Angular acceleration is the change in angular velocity in unit time:

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\varphi}}{dt^2}. \tag{1.19}$$

The total linear acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left[\vec{\omega},\vec{r}\right]}{dt} = \left[\frac{d\vec{\omega}}{dt},\vec{r}\right] + \left[\vec{\omega},\frac{d\vec{r}}{dt}\right] = \left[\vec{\beta},\vec{r}\right] + \left[\vec{\omega},\left[\vec{\omega},\vec{r}\right]\right], \tag{1.20}$$

where $\vec{a}_\tau = \left[\vec{\beta},\vec{r}\right]$ is the tangential acceleration, $\left|\vec{a}_\tau\right| = \beta R$;

$\vec{a}_r = \left[\vec{\omega},\left[\vec{\omega},\vec{r}\right]\right]$ is the radial acceleration, $\left|\vec{a}_r\right| = \omega^2 R$.

Period of rotation is the time interval required for the body to travel one complete circle. In the time interval T, the body moves a distance of $2\pi R$ and its angular displacement is 2π.

$$T = \frac{2\pi R}{v} = \frac{2\pi}{\omega}. \tag{1.21}$$

Frequency of the circular motion:

$$\upsilon = \frac{1}{T} = \frac{v}{2\pi R} = \frac{\omega}{2\pi} \tag{1.22}$$

Questions for self-control

1. What is the subject of kinematics?
2. Give definition to the point mass and rigid body
3. What does the reference frame consist of?
4. What is the difference between path and displacement?
5. Give definitions to the velocity and speed
6. Formulate the velocity addition law.
7. Definition of the acceleration and its relations with velocity and displacement
8. Equations for position vector and velocity for the case of uniformly accelerated motion
9. Particularities of acceleration of an object under curvilinear motion.
10. Projectile motion and its characteristics
11. Characteristics of the rotational motion.

Topic 1.2 Dynamics of translational motion

Lecture 2

Dynamics is the branch of classical mechanics which studies the motion of bodies with regard to the reasons which cause this motion.

An influence which causes a free body to move and accelerate is action of forces or torques. According to this dynamics falls under two categories: linear (studies effects of forces on the body's motion) and rotational (studies effects of torques on the body's motion)

1.2.1 Linear Dynamics

Inertial reference frames are the frames of reference which are in a state of rest or uniform, rectilinear motion with respect to each other.

Inertia is the resistance of any physical object to any change in its state of motion: it is the tendency of objects to keep moving in a straight line at constant velocity. The greater is the inertia of a body, the less acceleration it receives under constant force.

Mass m is the property of an object that specifies how much resistance an object exhibits to changes in its velocity. It is the measure of inertia of the body. Mass is an inherent property of the body due to the amount of substance it contains. It is a scalar quantity and thus obeys the rules of ordinary arithmetic.

Force $\vec{F}$ is any interaction that changes the motion of an object.

Force causes an object with mass to change its velocity, i.e., to accelerate. Force has vector nature.

All forces fall into two categories: *contact forces*, which involve physical contact between two objects (e.g. forces of friction, tension), and *field forces*, which do not involve physical contact between two objects but are due to the fields created by these objects (e.g. gravitational force, electromagnetic force).

If the body is under action of several forces, a **<u>net force</u>** is the vector sum of all the forces acting on this body. In linear dynamics, it has the same effect on the particle or rigid body as the original system of forces.

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n \qquad (1.23)$$

<u>Momentum</u> is the product of the mass and velocity of an object: $\vec{p} = m\vec{v}$.

1.2.2 Newton's laws of motion

<u>Newton's laws of motion</u> are three physical laws that, together, lay the foundation for classical mechanics.

Newton's first law (law of inertia):

When viewed from an inertial reference frame, an object at rest remains at rest and an object in motion continues in motion with a constant velocity (that is, with a constant speed in a straight line), unless acted upon by an external (unbalanced) force.

In other words, when no external force acts on an object, the acceleration of the object is zero.

Newton's second law:

In an inertial reference frame, the vector sum of the forces acting on an object is equal to the mass *m* of that object multiplied by the acceleration vector $\vec{a}$ of the object:

$$\vec{F} = m\vec{a} \,, \qquad (1.24)$$

where $\vec{F} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n$ is the net force. This equation is called the **<u>equation of motion</u>** of the object.

In other words, the acceleration of an object is directly proportional to the net force applied to it and inversely proportional to its mass:

$$\vec{a} = \frac{\sum_{i=1}^{n} \vec{F}_i}{m} \,. \qquad (1.25)$$

Equivalent formulation: the rate of change of the momentum of a body is directly proportional to the net force applied and this change in momentum takes place in the direction of the net force.

$$\vec{F} = \frac{d\vec{p}}{dt}, \tag{1.26}$$

where $d\vec{p}$ is the change in the momentum over the time interval dt.

<u>Impulse</u> $\vec{F}dt = d\vec{p}$. Impulse applied to an object produces an equivalent vector change in its linear momentum, also in the same direction.

Newton's third law (action-reaction law):

If two objects interact, the force $\vec{F}_{12}$ exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force $\vec{F}_{21}$ exerted by object 2 on object 1.

$$\vec{F}_{12} = -\vec{F}_{21}, \tag{1.27}$$

where $\vec{F}_{12}$ is action force, $\vec{F}_{21}$ is reaction force. The action and the reaction are simultaneous, they act on different objects and are of the same nature.

1.2.3 Fundamental mechanical forces

The fundamental forces laying in the basis of all the mechanical phenomena are the gravitational force and the electrostatic force.

<u>Newton's law of universal gravitation</u>: a particle attracts every other particle in the universe using a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them:

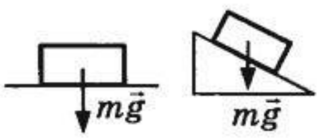
$$F_1 = F_2 = G\frac{m_1 m_2}{r^2}, \tag{1.28}$$

where F_1, F_2 are magnitudes of gravitational forces; G is the gravitational constant, $G = 6.67 \cdot 10^{-11} \frac{Nm^2}{kg^2}$; m_1, m_2 are **<u>gravitational masses</u>** of the bodies, gravitational mass and inertial mass have the same value; *r* is the distance between the centers of the masses.

The force of gravity is directed along a line joining the centers of masses of the objects.

At the Earth's surface, attractive force exerted by the Earth on an object of mass *m* is called the **gravitational force**

$$\vec{F} = m\vec{g} \tag{1.29}$$



Gravitational force is the reason of the free-fall of the bodies and gives them the acceleration due to gravity $\vec{g}$. It is directed vertically downward.

At the Earth's surface $mg = G\dfrac{Mm}{R^2}$, where *M* is the mass of the Earth, *R* is the radius of the Earth. So, $g = G\dfrac{M}{R^2} \approx 9.8 m/s^2$.

Gravitational acceleration $\vec{g}$ depends on the height *h* above the Earth's surface:

$$g = G\frac{M}{(R+h)^2}, \tag{1.30}$$

where *M* is the mass of the Earth (or another planet); *R* is the radius of the planet; *h* is the height above the planet's surface.

The first cosmic speed (the orbital speed) is the minimum speed required for the object to become a satellite of the planet, i.e. to start moving in the circular orbit around the planet.

According to the Newton's second law, $ma = m\dfrac{v^2}{R} = G\dfrac{Mm}{R^2}$, and

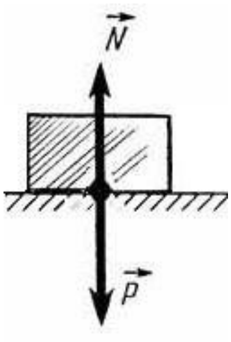
$$v_1 = \sqrt{G\frac{M}{R}} \approx 7.9 \text{ km/s}. \tag{1.31}$$

The second cosmic speed (the escape speed) is the minimum speed the object must have at the Earth's surface to approach an infinite separation distance from the Earth, i.e. to escape from the gravitational influence of the planet.

The magnitude of the escape speed can be found from the condition that the body's kinetic energy must be equal to its gravitational potential energy near the Earth's surface, as far as its gravitational potential energy on the height h approaches zero when h approaches infinity.

$$v_2 = \sqrt{2G\frac{M}{R}} \approx 11.16 \text{ km/s}. \qquad (1.32)$$

1.2.4 Non-fundamental mechanical forces



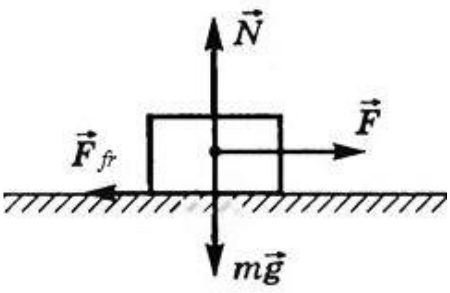
Weight $\vec{P}$ of a body is the force exerted by the body on the surface or on the string due to gravity.

Normal force $\vec{N}$ is the force acting at the point of contact of two objects in a direction normal to the surface interface between these two objects.

$\vec{P} = -\vec{N}$.

Normal force is the reason keeping the contacting objects separated due to repulsive forces of interaction between atoms at close contact.

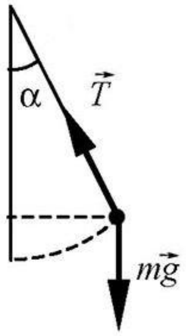
When the object is moving with acceleration $\vec{a} \uparrow\uparrow \vec{g}$, then according to the Newton's second law, $m\vec{a} = m\vec{g} + \vec{N}$; $ma = mg - N$; $N = m(g - a)$. If $a = g$, then $N = 0$ and the weight of the object equals zero: this is the state called **weightlessness**.



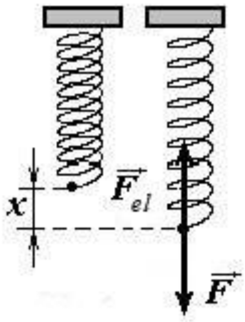
Friction is the force resisting the relative motion of objects sliding against each other. It is directed along the interface between these two objects opposite to the direction of motion.

$F_{fr} = \mu N$, where μ is the coefficient of friction,

N is the normal force.



Tension force is the pulling force exerted by a rope on the object in a direction away from the object, parallel to the rope. The magnitude of that force is called the tension in the rope.



Elastic force is the force acting to return a deformed spring to its natural length. According to the **Hooke's law**, the force exerted by an ideal spring equals:

$\vec{F}_{el} = -k\vec{x}$, where k is the **spring constant** – the measure of the stiffness of the spring,

$\vec{x}$ is the displacement of the spring.

1.2.5 Application of the fundamental equation of dynamics

The fundamental equation of dynamics is the equation of motion of the object:

$$m\frac{d\vec{v}}{dt} = \sum_{i=1}^{n} \vec{F}_i . \qquad (1.33)$$

Solvation of this equation is the fundamental problem of dynamics. First, a force diagram showing all the forces acting on the object should be made. It is called a **free-body diagram**. In a free-body diagram, the particle model is used by representing the object as a dot and showing the forces that act on the object as being applied to the dot

(Figure 1.9). A free-body diagram helps us isolate only those forces on the object and eliminate the other forces from our analysis.

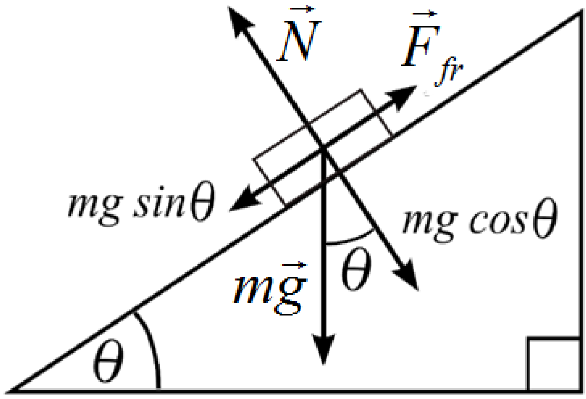


Figure 1.9 Example of the free-body diagram for a body on an inclined plane.

The following procedure is recommended when dealing with problems involving Newton’s laws:

1. Conceptualize. Draw a simple, neat diagram of the system. Establish convenient coordinate axes for each object in the system.

2. Categorize. If an acceleration component for an object is zero, the object is modeled as a particle in equilibrium in this direction and $\sum_{i=1}^{n}\vec{F}_i = 0$. If not, the object is modeled as a particle under a net force in this direction and $\sum_{i=1}^{n}\vec{F}_i = m\vec{a}$.

3. Analyze. Isolate the object whose motion is being analyzed. Draw a free-body diagram for this object. For systems containing more than one object, draw separate free-body diagrams for each object. Do not include in the free-body diagram forces exerted by the object on its surroundings.

Find the components of the forces along the coordinate axes:

$$\sum_{i=1}^{n} F_{ix} = ma_x; \qquad \sum_{i=1}^{n} F_{iy} = ma_y; \qquad \sum_{i=1}^{n} F_{iz} = ma_z.$$

Apply the appropriate model from the Categorize step for each direction. Check your dimensions to make sure that all terms have units of force.

Solve the component equations for the unknowns. Remember that you generally must have as many independent equations as you have unknowns to obtain a complete solution.

4. Finalize. Make sure your results are consistent with the free-body diagram. Also check the predictions of your solutions for extreme values of the variables. By doing so, you can often detect errors in your results.

1.2.6 Noninertial reference frames

A **non-inertial reference frame** is a frame of reference that is undergoing acceleration with respect to an inertial frame.

To explain the motion of bodies entirely within the viewpoint of non-inertial reference frames, **fictitious forces** (also called inertial forces, pseudo-forces) must be introduced to account for the observed motion. Fictitious force is due to observations made in an accelerated reference frame. A fictitious force appears to act on an object in the same way as a real force.

Let's consider the inertial reference frame *S* and the noninertial reference frame *S'* in the most general case, when the frame *S'* is under translational motion with acceleration $\vec{a}_0$ and rotation with constant angular velocity ω relative to the frame *S*. Then, according to the Galilean velocity transformation law, the velocity of an object

$\vec{v} = \vec{v}' + \vec{v}_0 + [\vec{\omega}, \vec{r}]$, where $\vec{v}'$ is velocity of the object relative to the frame *S'*, $\vec{v}_0$ is velocity of translational motion of the frame *S'* relative to the frame *S*, $[\vec{\omega}, \vec{r}]$ is the linear velocity of the frame *S'* due to its rotation relative to the frame *S*. The acceleration

$$\vec{a} = \frac{d\vec{v}'}{dt} + \frac{d\vec{v}_0}{dt} + \left[\vec{\omega}, \frac{d\vec{r}}{dt}\right], \qquad (1.34)$$

where $\frac{d\vec{v}_0}{dt} = \vec{a}_0$;

as far as in the frame *S'* the object is under rotational motion relative to the frame *S*, for its velocity

$$\frac{d\vec{v}'}{dt} = \frac{d[\vec{\omega}, \vec{r}']}{dt} = \left[\frac{d\vec{\omega}}{dt}, \vec{r}'\right] + \left[\vec{\omega}, \frac{d\vec{r}'}{dt}\right] = \vec{a}' + [\vec{\omega}, \vec{v}'];$$

and $\left[\vec{\omega},\frac{d\vec{r}}{dt}\right]=\left[\vec{\omega},(\vec{v}\,'+\left[\vec{\omega},\vec{r}\right])\right]=\left[\vec{\omega},\vec{v}\,'\right]+\left[\vec{\omega},\left[\vec{\omega},\vec{r}\right]\right]$

So, $\vec{a}=\vec{a}\,'+\vec{a}_0+2\left[\vec{\omega},\vec{v}\,'\right]+\left[\vec{\omega},\left[\vec{\omega},\vec{r}\right]\right]$, or $\vec{a}=\vec{a}\,'+\vec{a}_0+2\left[\vec{\omega},\vec{v}\,'\right]-\omega^2R\vec{n}$, where R is radius of the circular trajectory, $\vec{n}$ is the normal vector. Then $\vec{a}\,'=\vec{a}-\vec{a}_0+2\left[\vec{v}\,',\vec{\omega}\right]+\omega^2R\vec{n}$, and $m\vec{a}\,'=m\vec{a}-m\vec{a}_0+m\omega^2R\vec{n}+2m\left[\vec{v}\,',\vec{\omega}\right]$.

We obtain the **fundamental equation of dynamics in the noninertial reference frame**:

$$\sum_{i=1}^{n}\vec{F}\,'_i=\sum_{i=1}^{n}\vec{F}_i-m\vec{a}_0+m\omega^2R\vec{n}+2m\left[\vec{v}\,',\vec{\omega}\right] \qquad (1.35)$$

Here

$\vec{F}_{tr}=-m\vec{a}_0$ is the **translational inertial force** on the object in the frame *S'* due to the translational motion of the frame *S'* with acceleration $\vec{a}_0$ relative to the frame *S*; it is directed opposite to the direction of $\vec{a}_0$.

$\vec{F}_{centr}=m\omega^2R\vec{n}$ is the **centrifugal inertial force** on the object in the frame *S'* due to rotation of the frame *S'* relative to the frame *S*; it is directed away from the axis of rotation along the radius;

$\vec{F}_{cor}=2m\left[\vec{v}\,',\vec{\omega}\right]$ is the **Coriolis inertial force** on the object in the frame *S'* due to the object's motion with velocity $\vec{v}\,'$ in the rotating reference frame *S'*; it is directed perpendicular to the object velocity $\vec{v}\,'$ (see Figure 10).

As far as $\vec{F}_{cor}\perp\vec{v}\,'$, that is perpendicular to the direction of the motion, it does not perform work but deviates the object which rotates without changing its velocity.

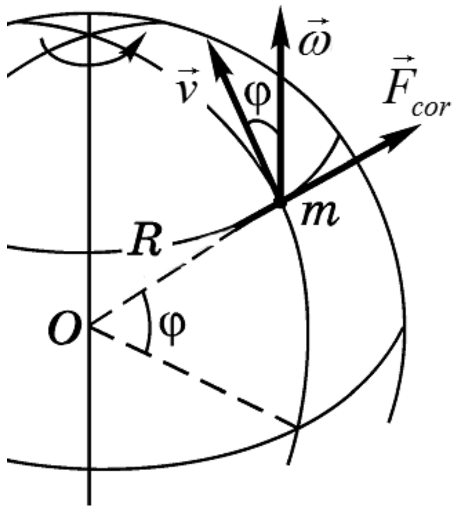


Figure 1.10 The Coriolis force on the particle of mass *m* and velocity $\vec{v}$ due to the rotation of the Earth.

1.2.7 Linear momentum of the system of particles

Linear momentum of a system of particles is the vector sum of their momenta:

$$\vec{p} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i \vec{v}_i \,, \tag{1.36}$$

where $\vec{p}_i$ is the linear momentum of the *i*-th particle.

According to the Newton's second law, $\frac{d\vec{p}_i}{dt} = \sum_{k=1}^{n} \vec{F}_{ik} + \vec{F}_i$, where $\vec{F}_{ik}$ are forces on the *i*-th particle exerted by the other particles of the system, $\vec{F}_i$ is net force from the environment acting on the *i*-th particle (external force). Then, $\frac{d\vec{p}}{dt} = \sum_{i,k=1}^{n} \vec{F}_{ik} + \sum_{i=1}^{n} \vec{F}_i$. The double sum $\sum_{i,k=1}^{n} \vec{F}_{ik}$ is the sum of all internal forces between the particles of the system. But according to the Newton's third law, $\vec{F}_{ik} = -\vec{F}_{ki}$, hence $\sum_{i,k=1}^{n} \vec{F}_{ik} = 0$.

So, the rate of change of the linear momentum of the system equals to the net external force acting on the system:

$$\frac{d\vec{p}}{dt} = \sum_{i=1}^{n} \vec{F}_i \,. \tag{1.37}$$

Only external forces can change the momentum of the system.

<u>**An isolated system**</u> is a physical system where no external forces are acting or the net external force on the system is zero. If $\sum_{i=1}^{n} \vec{F}_i = 0$ then $\frac{d\vec{p}}{dt} = 0$, consequently $\vec{p} = const$.

<u>**Law of conservation of linear momentum**</u>: in the isolated system the total linear momentum of a system of particles is conserved in time:

$$\vec{p} = \sum_{i=1}^{n} \vec{p}_i = const \,. \tag{1.38}$$

1.2.8 Motion of the system with variable mass

A system, mass of which is constantly changing as a result of accession or separation of its components, is called a system with variable mass. The motion of such system is called **propulsion** and can be considered in terms of the linear momentum conservation law. For example, when a rocket moves in free space, its linear momentum changes when some of its mass is ejected in the form of exhaust gases. Because the gases are given momentum when they are ejected out of the engine, the rocket receives a compensating momentum in the opposite direction. Therefore, the rocket is accelerated as a result of the "push," or thrust, from the exhaust gases.

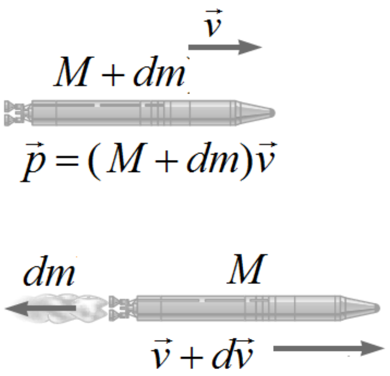


Figure 1.11 Rocket propulsion.

Suppose at some time *t* the magnitude of the momentum of a rocket plus its fuel is $(M + dm)\vec{v}$, where $\vec{v}$ is the velocity of the rocket relative to the Earth (see Figure 1.11). Over a short time interval *dt*, the rocket ejects fuel of mass *dm*. At the end of this interval, the rocket's mass is *M* and its velocity is $\vec{v} + d\vec{v}$, where $d\vec{v}$ is the change in the velocity of the rocket.

If the fuel is ejected with a velocity $\vec{u}$ relative to the rocket (the exhaust velocity), the velocity of the fuel relative to the Earth is $\vec{v} - \vec{u}$. Because the system of the rocket and the ejected fuel is isolated, we can equate the total initial momentum of the system to the total final momentum and obtain

$$(M + dm)\vec{v} = M(\vec{v} + d\vec{v}) + dm(\vec{v} - \vec{u}). \qquad (1.39)$$

Simplifying this expression gives $Md\vec{v} = dm \cdot \vec{u}$. Furthermore, the increase in the exhaust mass *dm* corresponds to an equal decrease in the rocket mass, so *dm* = – *dM*. By dividing this equation by *dt* we obtain the law of motion of the system with variable mass:

$$M\vec{a} = \vec{u}\frac{dm}{dt} = -\vec{u}\frac{dM}{dt}. \qquad (1.40)$$

It can be interpreted as the force that occurs as a result of the change in mass, which is called the **jet force**. The **thrust** on the rocket is the force exerted on it by the ejected exhaust gases:

$$Thrust = \left| \vec{u}\,\frac{dM}{dt} \right| \tag{1.41}$$

If the system is not isolated, the equation of motion of the system with variable mass will be

$$M\vec{a} = \vec{F} - \vec{u}\,\frac{dM}{dt}, \tag{1.42}$$

where $\vec{F}$ is the net external force on the system (in the case of the rocket it may include gravity, air resistance force).

Let's obtain the expression for the velocity of the rocket propulsion.

$$M\frac{d\vec{v}}{dt} = -\vec{u}\,\frac{dM}{dt}; \qquad d\vec{v} = -\vec{u}\,\frac{dM}{M}; \qquad \int_{v_0}^{v} d\vec{v} = -\vec{u}\int_{M_0}^{M}\frac{dM}{M};$$

$$\vec{v} = \vec{v}_0 + \vec{u}\ln\frac{M_0}{M}. \tag{1.43}$$

Questions for self-control

1. Classical mechanics and limits of its use.
2. The concept of force, mass, momentum of the body.
3. Give definitions to inertial and non-inertial reference frames.
4. Formulate the first, second, third Newton's laws.
5. Fundamental and non-fundamental mechanical forces
6. Weight and weightlessness.
7. Fundamental equation of dynamics for translational motion.
8. Fundamental equation of dynamics in non-inertial reference frames
9. Law of conservation of linear momentum.
10. Jet motion.

Topic 1.3. Energy and work

Lecture 3

1.3.1 Mechanical work

Let us consider the situation, where the object (the system) undergoes a displacement along some trajectory while acted on by a force $\vec{F}$ that makes an angle α with the direction of the displacement.

Action of the force $\vec{F}$ over the elementary displacement $d\vec{r}$ is characterized by the elementary **work**, which is the scalar product of the force and displacement (see Figure 1.12):

$$dA = \vec{F} \cdot d\vec{r} = F \cos\alpha dr \, . \tag{1.44}$$

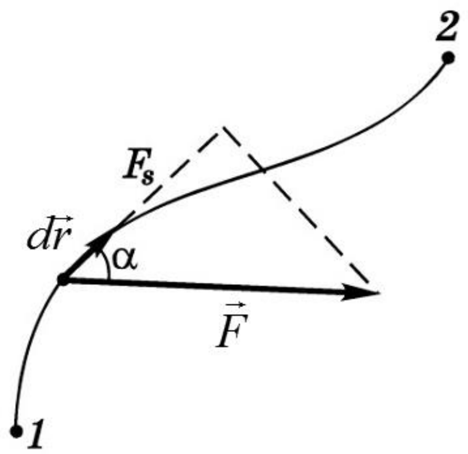


Figure 1.12.

The total work done for the displacement from r_1 to r_2 is

$$A = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = \int_{r_1}^{r_2} F \cos\alpha dr \, . \tag{1.45}$$

Work done by a constant force is $A = Fs\cos\alpha$, where F is the force applied over a displacement S, α is an angle between the vectors of force and displacement.

Work by the elastic force: $dA = -k\vec{r}d\vec{r} = -krdr$;

$$A_{el} = -\int_{r_1}^{r_2} krdr = \frac{kr_1^2}{2} - \frac{kr_2^2}{2} \tag{1.46}$$

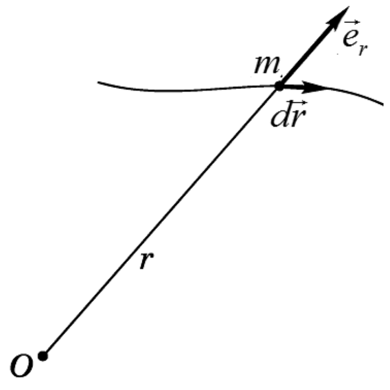


Figure 1.13

Work by the gravitational force:

$dA = G\dfrac{m_1 m_2}{r^2}\vec{e}_r d\vec{r} = G\dfrac{m_1 m_2}{r^2}dr$, where $\vec{e}_r$ is the unit radius-vector, dr is projection of $d\vec{r}$ on the radius-vector (see Figure 1.13).

$$A_{gr} = \int_{r_1}^{r_2} G\frac{m_1 m_2}{r^2}dr = Gm_1 m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad (1.47)$$

At the surface of the Earth, $dA = -mg\vec{e}_z d\vec{r} = -mgdz$, where $\vec{e}_z$ is the unit vector in the direction of the vertical axis OZ.

$$A_{gr} = -\int_{h_1}^{h_2} mgdz = mg(h_1 - h_2) \qquad (1.48)$$

A particle is said to be in the **field of forces** if at any point in space where the particle is placed the force is acting upon it. The field which is invariant with time is called stationary.

Conservative forces are forces of stationary field with these two equivalent properties:

1. The work done by a conservative force on a particle moving between any two points is independent of the path taken by the particle and depends only on its initial and final position.

2. The work done by a conservative force on a particle moving through any closed path is zero. (A closed path is one for which the beginning point and the endpoint are identical.)

The field of forces is called **central** if the force depends only on the distance between the interacting particles and is directed along the line joining these particles (for example, gravitational, elastic forces). *Central forces are conservative*. Friction force is nonconservative because its work depends on the length of the trajectory.

An important consideration is that work is an energy transfer. If work is done on a system by external agent, it is positive and energy is transferred to the system; if work is done by the system, it is negative and energy is transferred from the system. Therefore, if a system interacts with its environment, this interaction can be described as a transfer of energy across the system boundary.

1.3.2 Potential energy

As far as in the field of conservative forces work on the particle does not depend on the trajectory and depends only on its current position, the work may be considered as a function of the particle's position vector $\vec{r}$. This function is called the **potential energy** of the particle in the field of forces:

$$U(\vec{r}) = -\int dA = -\int \vec{F} \cdot d\vec{r} \qquad (1.49)$$

(when the work is done, potential energy decreases)

Potential energy is energy of interaction of a body and a force field, which is possessed by a body by virtue of its position relative to the other bodies or mutual position of the parts of the body.

The elastic potential energy is the energy stored in the deformed spring

$U_{el} = \frac{kx^2}{2}$, where k is the stiffness constant of the spring; x is the deformation of the spring.

The gravitational potential energy possessed by an elevated object near the Earth's surface due to the gravity

$U_{gr} = mgz$, where m is the mass of the body; z is the altitude of the body.

The gravitational potential energy of interaction of two objects of masses m_1, m_2

$U_{gr} = G\frac{m_1 m_2}{r}$, where r is the distance between the two objects.

In general, $dU(\vec{r}) = -\vec{F}\cdot d\vec{r} = -F_r dr$, where F_r is projection of the force $\vec{F}$ on the direction of the vector $d\vec{r}$. Then,

$F_r = -\dfrac{dU}{dr}$. Similarly, $F_x = -\dfrac{dU}{dx}; F_y = -\dfrac{dU}{dy}; F_z = -\dfrac{dU}{dz}$, and

$$\vec{F} = -\left(\frac{dU}{dx}\vec{i} + \frac{dU}{dy}\vec{j} + \frac{dU}{dz}\vec{k}\right) = -grad(U) \qquad (1.50)$$

Force of the field is equal to the negative gradient of the field potential energy.

Work by conservative force:

$$dA = -dU, \qquad (1.51)$$

$A = U_2 - U_1$, where U_1 is the initial value of the potential energy of the body; U_2 is the final value of the potential energy of the body

1.3.3 Kinetic energy

Kinetic energy of an object is the energy that it possesses due to its motion.

Consider a particle moving under the action of a net force $\vec{F}$ in the direction of that force. According to the Newton's second law, $\vec{F} = m\dfrac{d\vec{v}}{dt}$. The work by this force on the particle

$$A = \int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} = \int_{r_1}^{r_2} m\frac{d\vec{v}}{dt}\cdot d\vec{r} = m\int_{v_1}^{v_2} \vec{v}d\vec{v} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2}. \qquad (1.52)$$

The quantity $K = \dfrac{mv^2}{2}$ is called the **kinetic energy** of the particle.

The Work–Kinetic Energy Theorem: the work done on a body by a net force acting on it equals the change in kinetic energy of the body.

$$A = K_2 - K_1, \qquad (1.53)$$

where K_1 is the initial kinetic energy and K_2 is the final kinetic energy of the body.

The particle may be under the action of internal conservative forces of the field and external forces $\vec{F} = \vec{F}_{cons} + \vec{F}_{ex}$. Then the change in its kinetic energy $\Delta K = A_{cons} + A_{ex}$. But $A_{cons} = -\Delta U$, and $\Delta(K+U) = A_{ex}$. As we see, the increase of the sum of kinetic and potential energy of the particle is caused by the work of external forces.

The total mechanical energy of an object is the sum of its potential and kinetic energy $E = K + U$.

So, $\Delta E = A_{ex}$.

If the external forces on the particle are absent or their work equals zero, then $A_{ex} = 0$ and

$$K + U = const. \tag{1.54}$$

This is the law of conservation of the total mechanical energy in the stationary field of conservative forces.

The law of conservation of energy: the total energy of an isolated system remains constant — it is conserved over time. Energy can neither be created nor destroyed; rather, it transforms from one form to another.

$$E = const \tag{1.55}$$

If the system is not isolated or internal nonconservative forces (friction, resistance) are present, then $E_2 - E_1 = A_{ex} + A_{non-cons}$

Work by friction:

$$A_{fr} = E_2 - E_1, \tag{1.56}$$

where E_1 is the initial value of the total energy of the body; E_2 is the final value of the total energy of the body. The work by friction or by fluid resistance is always negative.

1.3.4 Power

Power is the rate of doing work. It is the amount of energy consumed per unit time. Power is a scalar quantity.

$$P = \frac{dA}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} \qquad (1.57)$$

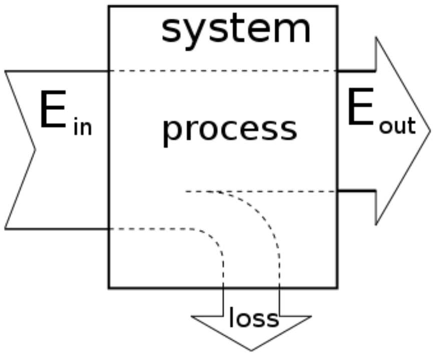
Instantaneous power:

$$P = \vec{F} \cdot \vec{v} = Fv\cos\alpha\,, \qquad (1.58)$$

where F is the force acting on the body; v is the magnitude of velocity of the body; α is the angle between the vectors of force and velocity.

Power of an engine:

$P = F_t v$, where F_t is the tractive force of the engine; v is the speed of the vehicle.



Energy conversion efficiency (η) is the ratio between the useful output of energy (power) and the consumed input of energy (power) of an energy conversion machine.

$$\eta = \frac{A_{out}}{A_{in}} = \frac{P_{out}}{P_{in}}\,, \qquad (1.59)$$

where A_{out}, P_{out} is the useful output of work (power); A_{in}, P_{in} is the total consumed input of work (power)

1.3.5 Collisions of two particles

Let's describe what happens when two particles collide. The two particles form an isolated system and the momentum of the system must be conserved in any collision. In contrast, the total kinetic energy of the system of particles may or may not be conserved, depending on the type of collision.

An **elastic collision** between two objects is one in which the total kinetic energy (as well as total momentum) of the system is the same before and after the collision.

Consider two particles of masses m_1 and m_2 moving with initial velocities $\vec{v}_{11}$ and $\vec{v}_{21}$. After the collision they move with velocities $\vec{v}_{12}$ and $\vec{v}_{22}$ (see Figure 1.14).

Then, according to the law of conservation of the linear momentum,

$$m_1\vec{v}_{11} + m_2\vec{v}_{21} = m_1\vec{v}_{12} + m_2\vec{v}_{22}$$

According to the law of conservation of the total mechanical energy of the system,

$$\frac{m_1 v_{11}^2}{2} + \frac{m_2 v_{21}^2}{2} = \frac{m_1 v_{12}^2}{2} + \frac{m_2 v_{22}^2}{2}.$$

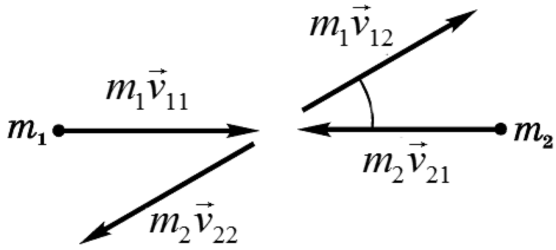


Figure 1.14 Collision of the two particles

To solve the problem involving collisions between two particles it is recommended to write expressions for the *x*, *y* and *z* components of the momentum of each object before and after the collision and to solve the obtained system of equations.

An **inelastic collision** is one in which the total kinetic energy of the system is not the same before and after the collision (even though the momentum of the system is conserved). Inelastic collisions are of two types.

– When the objects stick together after they collide (for example, a meteorite collides with the Earth), the collision is called **perfectly inelastic**.
– When the colliding objects do not stick together but some kinetic energy is transformed or transferred away (for example a rubber ball colliding with a hard surface), the collision is called **inelastic**.

Consider two particles of masses m_1 and m_2 moving with initial velocities $\vec{v}_{11}$ and $\vec{v}_{21}$. After the perfectly inelastic collision they stick together, and then move with some common velocity $\vec{v}$.

Then, according to the law of conservation of the linear momentum,

$$m_1\vec{v}_{11} + m_2\vec{v}_{21} = (m_1 + m_2)\vec{v};$$

$$\vec{v} = \frac{m_1\vec{v}_{11} + m_2\vec{v}_{21}}{m_1 + m_2}.$$

Questions for self-control

1. Definition of the mechanical work
2. What is the force field/
3. Conservative forces.
4. Definition of the potential energy
5. Gravitational potential energy
6. Elastic potential energy
7. Kinetic energy.
8. The work-kinetic energy theorem
9. Law of conservation of the total mechanical energy.
10. Collisions of two bodies.
11. Power

Topic 1.4 Dynamics of rotational motion

Lecture 4

1.4.1 Fundamentals of rotational dynamics

Rotational dynamics studies objects that are rotating or moving in a curved path. The reason of rotational motion of the object is action of applied torques.

Torque, or **moment of force** is the tendency of a force to rotate an object about an axis.

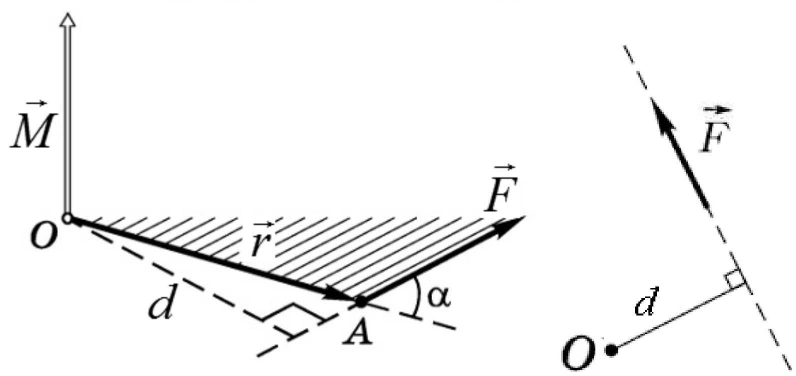
Torque about the point *O* is equal to the vector product of the radius vector, drawn from the point *O* to the point of application of the force, and the force:

$$\vec{M} = \left[\vec{r}, \vec{F}\right]. \tag{1.60}$$

Torque is the axial vector. Its magnitude

$$M = Fr\sin\alpha = Fd\,, \tag{1.61}$$

where *F* is the applied force, *α* is the angle between vectors $\vec{r}$ and $\vec{F}$, *d* is the length of the lever arm (see Figure 1.15).



Lever arm is the perpendicular distance from the axis of rotation to the line of action of the force.

Figure 1.15

Angular momentum of the particle about the point *O* is the vector product of the radius vector, drawn from the point *O* to that particle, and the linear momentum of this particle:

$$\vec{L} = [\vec{r}, \vec{p}]. \tag{1.62}$$

Angular momentum is the axial vector. Its magnitude

$$L = pr\sin\alpha = pd\ . \tag{1.63}$$

Let's consider a change in the angular momentum of the object:

$$\frac{d\vec{L}}{dt} = \left[\frac{d\vec{r}}{dt}, \vec{p}\right] + \left[\vec{r}, \frac{d\vec{p}}{dt}\right].$$

The first term of the sum equals zero because vectors $\vec{r}$ and $\vec{p}$ are parallel, while $\frac{d\vec{p}}{dt}$ equals the net force on the object according to the Newton's second law. So,

$$\frac{d\vec{L}}{dt} = \left[\vec{r}, \vec{F}\right] = \vec{M}\ , \tag{1.64}$$

where $\vec{M}$ is the net torque. This is the **<u>fundamental equation of dynamics of rotational motion</u>**. Torque causes the angular momentum to change just as force causes linear momentum to change.

The total angular momentum of a system of particles about some axis is defined as the vector sum of the angular momentums of the individual particles:

$$\vec{L}_{tot} = \sum_{i=1}^{n} \vec{L}_i\ . \tag{1.65}$$

The torques acting on the particles of the system are those associated with internal forces between particles and those associated with external forces. The net torque associated with all internal forces, however, is zero, as follows from the Newton's third law.

The total angular momentum of a system can vary with time only if a net external torque is acting on the system:

$$\frac{d\vec{L}_{tot}}{dt} = \vec{M}_{ex}\,. \tag{1.66}$$

On the other hand, the total angular momentum of a system is constant in both magnitude and direction if the net external torque acting on the system is zero, that is, if the system is isolated.

<u>The law of conservation of the angular momentum</u> in the isolated system:

$$\sum_{i=1}^{n} \vec{L}_i = const \qquad (1.67)$$

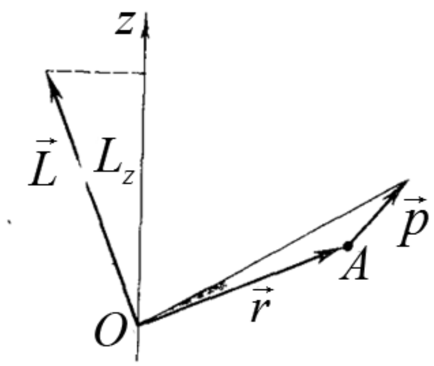


Figure 1.16 Angular momentum about the axis

<u>Angular momentum L_z about an axis</u> is the projection onto this axis of the angular momentum vector $\vec{L}$ found about the arbitrary point *O* laying on the axis *z* (see Figure 1.16).

<u>Torque M_z about an axis</u> is the projection onto this axis of the torque $\vec{M}$ found about the arbitrary point *O* laying on the axis *z*.

1.4.2 Conditions of equilibrium of the body:

$$\vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \sum_{i=1}^{n} \vec{F}_i = 0, \qquad M_1 + M_2 + ... + M_n = \sum_{i=1}^{n} M_i = 0, \qquad (1.68)$$

where $\sum_{i=1}^{n} \vec{F}_i$ is the vector sum of forces acting on the body; $\sum_{i=1}^{n} M_i$ is the algebraic sum of torques acting on the body considering their signs.

<u>Center of mass</u> is the unique point of balance of the distributed mass of the system of particles or the body.

Coordinates of the center of mass:

$$x_c = \frac{m_1 x_1 + m_2 x_2 + ... + m_n x_n}{m_1 + m_2 + ... + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}, \quad y_c = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}, \quad z_c = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i} \qquad (1.69)$$

where x_c, y_c, z_c are coordinates of the center of mass of the system of particles; x_i, y_i, z_i are coordinates of the masses of particles in the system; m_i are masses of particles in the system.

1.4.3 Dynamics of the rotating rigid object

1.4.3.1 Rotation about the fixed axis

Consider a rigid object rotating about a fixed axis that coincides with the *z* axis of a coordinate system (Figure 1.17). Then each particle of the object rotates in the *xy* plane about the *z* axis with an angular speed ω. For the *i*-th particle of the body, its angular momentum about the axis

$L_{zi} = R_i p_i$, where R_i is the radius of the circle traced out by the particle; momentum $p_i = m_i v_i = m_i \omega R_i$.

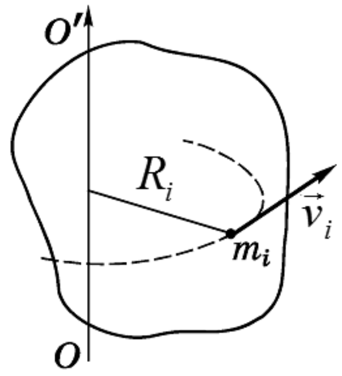


Figure 1.17.

Then, the angular momentum about the axis of the whole object will be

$$L_z = \sum_{i=1}^{n} L_{zi} = \sum_{i=1}^{n} m_i R_i^2 \omega = \omega\left(\sum_{i=1}^{n} m_i R_i^2\right). \qquad (1.70)$$

The quantity $I_z = \left(\sum_{i=1}^{n} m_i R_i^2\right)$ is called the **moment of inertia** of the object about the *z* axis. Then,

$$L_z = I_z \omega. \qquad (1.71)$$

Let's differentiate the last equation with respect to time, noting that I_z is constant for a rigid object:

$\frac{dL_z}{dt} = I_z \frac{d\omega}{dt} = I_z \beta$. Also, we know that the change in the angular momentum of the system of particles $\frac{dL_z}{dt} = M_z^{ex}$. So we obtain another formulation of the **fundamental equation of dynamics of rotational motion**:

$$M_z^{ex} = I_z \beta, \qquad (1.72)$$

where M_z^{ex} is the net external torque about the *z* axis, β is the angular acceleration.

It is the rotational form of Newton's second law. The net external torque acting on a rigid object rotating about a fixed axis equals the moment of inertia about the rotation axis multiplied by the object's angular acceleration relative to that axis.

Let us calculate the moment of inertia of rigid body in general case of continuous distribution of the volumetric mass:

$$I = \int r^2 dm = \int r^2 \rho dV, \tag{1.73}$$

where dm and dV are mass and volume of the elementary part of the body rotating in circle of radius r about the axis, ρ is volumetric density of the body.

As we see, the moment of inertia of the body depends on its mass distribution with regard to the axis of rotation. Let's find moment of inertia for some bodies of simple shape.

1.4.3.2 Moment of inertia of a rod with the mass m and the length l about the axis passing through its center of mass perpendicular to the rod

First we differentiate the rod into elementary masses dm which are located at distances x from the axis of rotation (Figure 1.18). The volume of every such component is $dV = dx \cdot S$, where S is the cross-sectional area of the rod; then $dm = \rho dV = \rho S dx$.

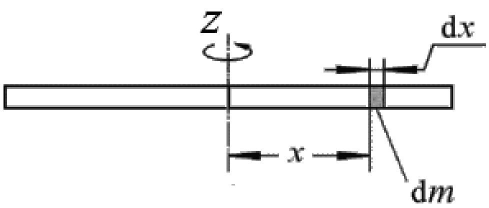


Figure 1.18.

After that we integrate:

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dm = \rho S \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dx = \frac{1}{12}\rho S l^3.$$

The mass of the rod is $m = \rho S l$, and we obtain

$$I = \frac{ml^2}{12}.$$

1.4.3.3 Moment of inertia of a uniform disc with the mass m and radius R about the axis passing through its center perpendicular to the plane of the disc

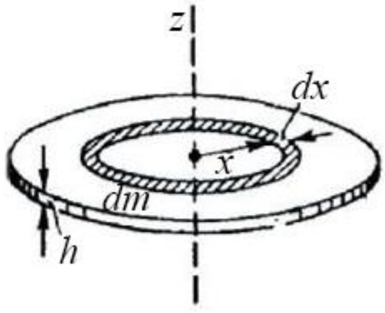


Figure 1.19

First we differentiate the disc into elementary masses dm which are thin rings of width dx and radius x (see Figure 1.19). The volume of such ring is $dV = 2\pi xhdx$, where h is the height of the disc. The mass of the ring is $dm = \rho dV = 2\pi xh\rho dx$. Since the disc is uniform, ρ is constant, and we can find the total moment of inertia as integral:

$$I = \rho\int_0^R x^2 2\pi xhdx = \frac{\rho\pi hR^4}{2}.$$

The mass of disk is $m = \rho\pi R^2 h$, and we obtain

$$I = \frac{mR^2}{2}.$$

1.4.3.4 Moment of inertia of a uniform sphere with the mass m and radius R about the axis passing through its center

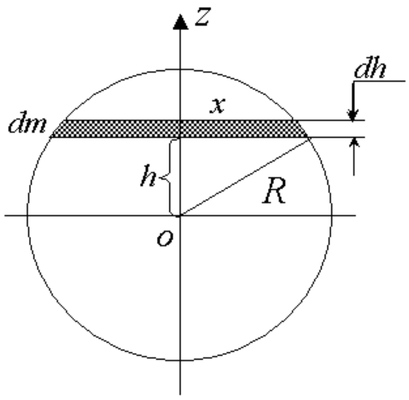


Figure 1.20.

First we differentiate the sphere into elementary masses dm which are thin disks of height dh and radius x (see Figure 1.20). The volume of such disk is $dV = \pi x^2 dh$. Radius x of the disk and distance h from the center of the sphere to the plane of the disk are related as $x^2 = R^2 - h^2$, then the mass of the disc $dm = \rho\pi(R^2 - h^2)dh$. As we have found previously, the moment of inertia of the disc of mass dm and radius x will be

$dI = \frac{x^2 dm}{2}$, and we can find the total moment of inertia of the sphere as integral, considering that ρ is constant:

$$I=\int_{-R}^{R}\frac{x^2dm}{2}=2\int_{0}^{R}\frac{(R^2-h^2)\rho\pi(R^2-h^2)dh}{2}=\pi\rho\int_{0}^{R}(R^2-h^2)^2dh=\frac{8\pi\rho R^5}{15}.$$

The mass of the sphere is $m=\frac{4}{3}\rho\pi R^3$, and we obtain

$$I=\frac{2mR^2}{5}.$$

The minimum value of the moment of inertia of the body is that found about the axis passing through its center of mass. The moment of inertia about the axis passing through the center of mass is called the **principal moment of inertia**.

1.4.3.5 The parallel axis theorem

Calculation of the moment of inertia of a body that has complex shape is quite a complicated problem but it may be simplified using the parallel axis theorem.

Parallel axis theorem (Steiner's theorem) can be used to determine the moment of inertia of a rigid body about any axis given the body's moment of inertia about a parallel axis through the object's center of mass, I_0, and the perpendicular distance between the axes, a:

$$I=I_0+ma^2 \tag{1.74}$$

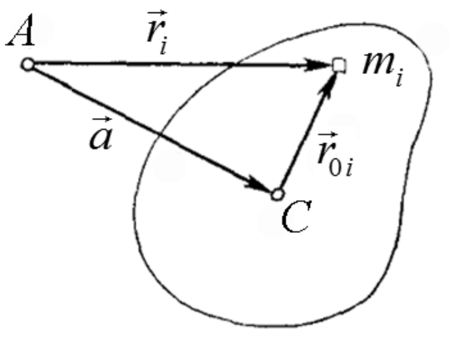


Figure 1.21.

Consider that the radius vector from the arbitrary axis A to the i-th particle of the body is $\vec{r}_i$, while the radius vector from the axis C passing through the center of mass to that particle is $\vec{r}_{0i}$ (see Figure 1.21). The perpendicular distance between the axes can be given by the vector $\vec{a}$. As we see, $\vec{r}_i=\vec{r}_{0i}+\vec{a}$. The moment of inertia of the entire body will be

$$I=\sum_{i=1}^{n}m_i\vec{r}_i^{\,2}=\sum_{i=1}^{n}m_i(\vec{r}_{0i}+\vec{a})^2=\sum_{i=1}^{n}m_i\vec{r}_{0i}^{\,2}+2\sum_{i=1}^{n}m_i\vec{r}_{0i}\vec{a}+\sum_{i=1}^{n}m_i\vec{a}^{\,2}, \text{ where}$$

$\sum_{i=1}^{n} m_i \vec{r}_{0i}{}^2 = I_0$ is the moment of inertia of the body about the axis through the body's center of mass;

$\sum_{i=1}^{n} m_i \vec{a}^2 = a^2 \sum_{i=1}^{n} m_i = ma^2$;

$2\sum_{i=1}^{n} m_i \vec{r}_{0i}\vec{a} = 2\vec{a}\sum_{i=1}^{n} m_i \vec{r}_{0i} = 0$ because $\sum_{i=1}^{n} m_i \vec{r}_{0i}$ is the position of the body's center of mass relative to the point *C*, which is the center of mass itself, so the distance between the center of mass and point *C* is zero.

In the result, we obtain the parallel axis theorem $I = I_0 + ma^2$.

1.4.3.6 Kinetic energy of the rotating rigid body:

Considering that speed of the *i*-th particle of the body $v_i = \omega R_i$, the kinetic energy of the whole body will be

$$K = \sum \frac{m_i v_i^2}{2} = \sum \frac{m_i (\omega R_i)^2}{2} = \left(\sum m_i R_i{}^2 \right) \frac{\omega^2}{2}, \text{ or}$$

$$K = \frac{I\omega^2}{2} \tag{1.75}$$

Work on the rotating rigid body:

According the work - kinetic energy theorem, work by external forces on the rotating rigid body equals to the increase in its kinetic energy:

$$dA = dK; \qquad dA = d\left(\frac{I\omega^2}{2} \right) = I\omega d\omega.$$

But, according to the fundamental equation of dynamics of rotational motion, $Id\omega = M_z dt$, then

$dA = M_z \omega dt = M_z \frac{d\varphi}{dt} dt$, where φ is the angle of rotation. We obtain

$$dA = M_z d\varphi; \qquad A = \int_{\varphi_1}^{\varphi_2} M_z d\varphi, \tag{1.76}$$

where M_z is the net torque by external forces on the body.

Questions for self-control

1. Definition of the rotational motion of a rigid body
2. Torque about a point and about an axis.
3. Angular momentum about a point and about an axis.
4. Law of conservation of angular momentum.
5. Fundamental equation of dynamics of rotational motion.
6. Conditions of equilibrium of the body.
7. Center of mass
8. Moment of inertia
9. Moment of inertia for bodies of different shape.
10. The parallel axis theorem
11. Kinetic energy of the rotating rigid body
12. Work on the rotating rigid body

Lecture 5

1.4.3.7 Rolling motion of a rigid body

Consider a rigid object rolling along a flat surface. **Rolling** is a type of motion that combines translational and rotational motion of that object with respect to a surface (for the *pure rolling* sliding is neglected): center of mass *C* of the object is under translational motion in some plane, while the object itself is rotating about the axis passing through the center of mass. In other words, axis of rotation of the object is under translational motion, and the object is under rotational motion relative to the reference frame, connected with that axis.

The rolling motion of the object can be described by the following equations:

$$m\vec{a}_C = \vec{F}\,; \qquad I_C\beta_z = M_{Cz}\,, \tag{1.77}$$

where $\vec{a}_C$ is acceleration of the center of mass, $\vec{F}$ is the net external force on the object, β_z is angular acceleration, I_C and M_{Cz} are moment of inertia and net external torque about the axis, passing through the center of mass.

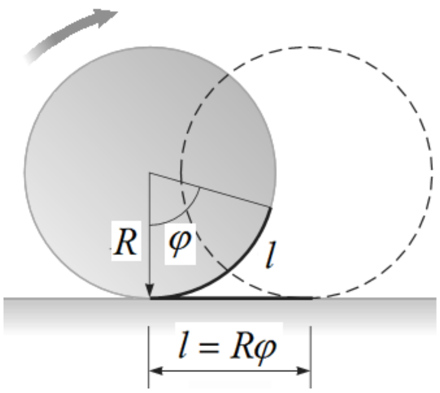


Figure 1.22.

As the object rotates through an angle φ, its center of mass moves a linear distance $l = \varphi R$ (see Figure 1.22). Therefore, the translational speed of the center of mass for pure rolling motion is given by $v_C = \dfrac{dl}{dt} = \dfrac{Rd\varphi}{dt} = R\omega_z$, where ω_z is the angular speed of rotation.

Velocity of the arbitrary *i*-th particle of the object can be found from the velocity –addition law as the sum of its linear velocity due to the object's rotation in the reference frame connected with the center of mass, and the velocity of the center of mass:

$$\vec{v}_i = \omega r_i \vec{\tau} + \vec{v}_C\,, \tag{1.78}$$

where r_i is the radius of the circle traced out by this particle, $\vec{\tau}$ is the unit vector tangent to the circle (linear velocity is directed tangent to the trajectory).

The magnitude of the linear acceleration of the center of mass for pure rolling motion is

$$a_C = \frac{dv_C}{dt} = \frac{Rd\omega_z}{dt} = R\beta_z. \tag{1.79}$$

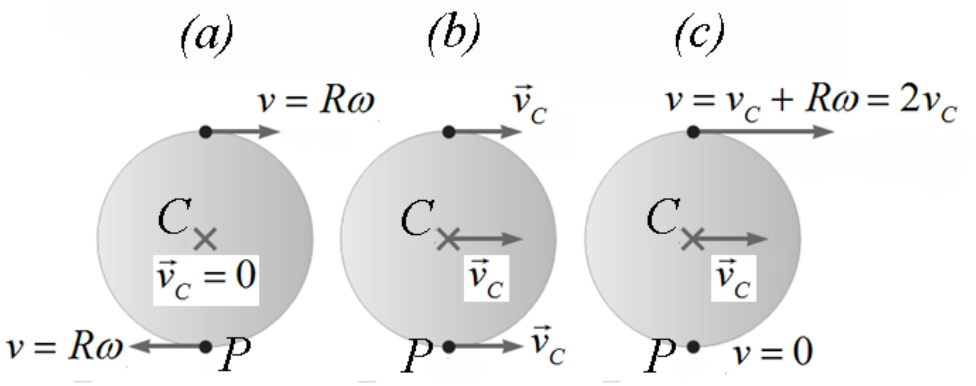


Figure 1.23:

(a) Pure rotation

(b) Pure translation

(c) Combination of translation and rotation

Total kinetic energy of the rolling object

Notice that the contact point *P* between the surface and object has a translational speed of zero $v_P = v_C - \omega R = \omega R - \omega R = 0$ (see Figure 1.23). So motion of the object may be considered as pure rotation about the fixed axis passing through the instantaneous point *P*, and its kinetic energy

$$K = \frac{I_p\omega^2}{2},$$

where I_p is the moment of inertia about the axis through the point *P*. Applying the parallel-axis theorem, $I_p = I_C + mR^2$, we obtain $K = \frac{I_C\omega^2}{2} + \frac{m(R\omega)^2}{2}$. Using that $R\omega = v_C$,

$$K = \frac{I_C\omega^2}{2} + \frac{mv_C{}^2}{2}. \tag{1.80}$$

1.4.3.8 Inertia tensor

Consider a free rotational motion of a rigid body when the axis of rotation is free to assume any orientation by itself. The motion of the body becomes complicated. Let's take the reference frame connected with the center of mass *C* of the body.

The *i*-th particle of mass m_i will have angular momentum about the point *C*

$$\vec{L}_{Ci} = \left[\vec{r}_i, \vec{p}_i\right] = \left[\vec{r}_i, m_i\vec{v}_i\right] = \left[\vec{r}_i, m_i\left[\vec{\omega}, \vec{r}_i\right]\right].$$

Using the rule of double vector product, $\left[\vec{A}\left[\vec{B}\vec{C}\right]\right] = \vec{B}\left(\vec{A}\vec{C}\right) - \vec{C}\left(\vec{A}\vec{B}\right)$, we obtain

$$\vec{L}_{Ci} = m_i\left(\vec{\omega}\vec{r}_i^{\,2} - \vec{r}_i\left(\vec{r}_i, \vec{\omega}\right)\right).$$

The angular momentum of the whole body is

$$\vec{L}_C = \sum \vec{L}_{Ci} = \sum m_i\left(\vec{\omega}r_i^2 - \vec{r}_i\left(\vec{r}_i, \vec{\omega}\right)\right). \tag{1.81}$$

As we see, direction of the angular momentum vector $\vec{L}_C$ does not coincide with the direction of angular velocity vector $\vec{\omega}$.

Let us write the scalar products in coordinate form:

$$\left(\vec{r}_i, \vec{\omega}\right) = x_i\omega_x + y_i\omega_y + z_i\omega_z, \qquad \vec{r}^{\,2} = x_i^2 + y_i^2 + z_i^2.$$

Then, projections of the vector $\vec{L}_C$ on the coordinate axes will be

$$L_{Cx} = \sum m_i\left(\left(x_i^2 + y_i^2 + z_i^2\right)\omega_x - x_i\left(x_i\omega_x + y_i\omega_y + z_i\omega_z\right)\right) =$$
$$= \sum m_i\left(\left(y_i^2 + z_i^2\right)\omega_x - x_i\left(y_i\omega_y + z_i\omega_z\right)\right) =$$
$$= \omega_x\sum m_i\left(y_i^2 + z_i^2\right) - \omega_y\sum m_i x_i y_i - \omega_z\sum m_i x_i z_i.$$

Similarly,

$$L_{Cy} = \omega_y\sum m_i\left(x_i^2 + z_i^2\right) - \omega_x\sum m_i x_i y_i - \omega_z\sum m_i y_i z_i;$$

$$L_{Cz} = \omega_z\sum m_i\left(x_i^2 + y_i^2\right) - \omega_x\sum m_i x_i z_i - \omega_y\sum m_i z_i y_i.$$

Let us introduce notations:

$$I_{xx} = \sum_{i=1}^{n} m_i \left(y_i^2 + z_i^2 \right); \quad I_{yy} = \sum_{i=1}^{n} m_i \left(x_i^2 + z_i^2 \right); \quad I_{zz} = \sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2 \right),$$

where I_{xx}, I_{yy}, I_{zz} are called **axial moments of inertia** with respect to the *x*, *y* and *z* axis.

$$I_{xy} = -\sum_{i=1}^{n} m_i x_i y_i = I_{yx}; \quad I_{xz} = -\sum_{i=1}^{n} m_i x_i z_i = I_{zx}; \quad I_{yz} = -\sum_{i=1}^{n} m_i y_i z_i = I_{zy},$$

where I_{xy}, I_{xz}, I_{yz} are called **products of inertia**, which are a measure of the imbalance in the mass distribution.

Consequently, coordinates of the angular momentum

$$L_{Cx} = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z;$$

$$L_{Cy} = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z;$$

$$L_{Cx} = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z.$$

Thus, mutual orientation of vectors $\vec{L}_C$ and $\vec{\omega}$ is defined by values of coefficients of proportionality between them. For example, if $I_{xx} = I_{yy} = I_{zz} = I$ and $I_{xy} = I_{yz} = I_{xz} = 0$, then $L_{Cx} = I\omega_x$; $L_{Cy} = I\omega_y$; $L_{Cz} = I\omega_z$. Vectors $\vec{L}_C$ and $\vec{\omega}$ in this case are collinear.

Then, the expression for angular momentum can be written in matrix form as

$$\begin{pmatrix} L_{Cx} \\ L_{Cy} \\ L_{Cz} \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \cdot \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}, \tag{1.82}$$

where $J = \begin{pmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{xx} \end{pmatrix}$ is the **tensor of inertia** (written in matrix form) about the center of mass *C* and with respect to the *xyz* axes. The tensor of inertia gives us an idea about how the mass is distributed in a rigid body.

For a general three-dimensional body, it is always possible to find 3 mutually orthogonal axis for which the products of inertia are zero, and the inertia matrix takes a diagonal form. Such axes are called **principal axes** of inertia. For the rotation about one of these axes, the angular momentum vector is parallel to the angular velocity vector. For symmetric bodies principle axes coincide with the axes of symmetry of the body. One of the principal axes is the axis of symmetry, while two other axes are perpendicular to it and pass through the center of mass of the body.

1.4.3.9 Motion of gyroscope

A **gyroscope** is a symmetrical rigid body rotating about its axis of symmetry at high angular speed. If the body spins rapidly, the symmetry axis rotates about the *z* axis, sweeping out a cone. The motion of the symmetry axis about the vertical axis is called the **precessional motion**.

The higher is the body's angular speed ω, the lesser is the angular speed of rotation of the symmetry axis ω_{pr}, and $\omega >> \omega_{pr}$ (condition of high angular speed). The angular momentum of the gyroscope is the sum of angular momentum about the axis of symmetry, *L*, and angular momentum about the *z* axis, *L'*. But due to the condition $\omega >> \omega_{pr}$, also $L >> L'$, and *L'* can be neglected. So, gyroscope's angular momentum equals its angular momentum about the axis of symmetry:

$$\vec{L} = I_C \vec{\omega}, \tag{1.83}$$

and vector $\vec{L}$ is directed along the symmetry axis.

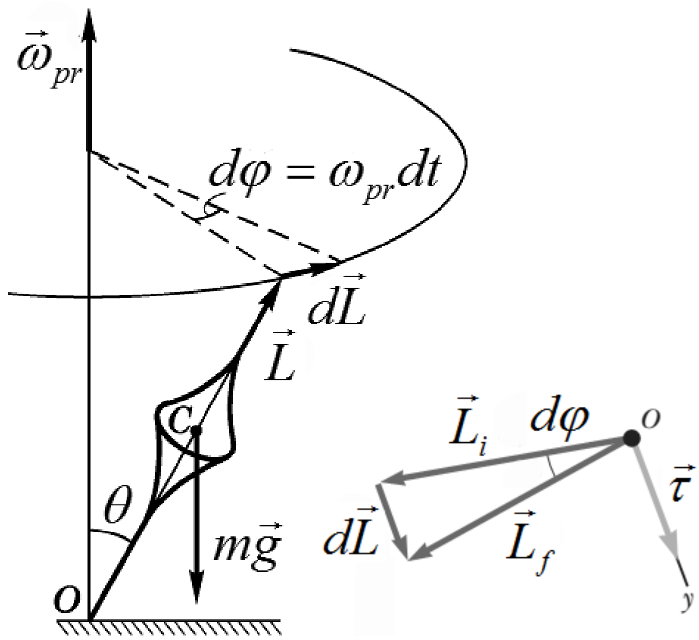


Figure 1.24.

The symmetry axis rotates about the z axis (precessional motion occurs) because the torque produces a change in the direction of the symmetry axis. Consider a simple gyroscope (top) shown in Figure 1.24. The two forces are acting on the gyroscope: the downward gravitational force $m\vec{g}$ and the normal force $\vec{N}$ acting upward at the pivot point O.

The normal force produces no torque about an axis passing through the pivot because its lever arm is zero. The gravitational force, however, produces a torque $\vec{M} = [\vec{r}, m\vec{g}]$ about an axis passing through O, where the direction of $\vec{M}$ is perpendicular to the plane formed by $\vec{r}$ and $m\vec{g}$. The vector $\vec{M}$ lies in a horizontal xy plane perpendicular to the angular momentum vector $\vec{L}$. The net torque and angular momentum of the gyroscope are related through equation $\frac{d\vec{L}}{dt} = \vec{M}$, where $\vec{M}$ is the net external torque on the body. Consequently, the vector of change in angular momentum $d\vec{L}$ must be in the same direction as the vector of net torque $\vec{M}$ producing it. Therefore, like the torque vector, $d\vec{L}$ must also be perpendicular to $\vec{L}$ (Figure 1.24).

Because $d\vec{L}$ is perpendicular to $\vec{L}$, the magnitude of $\vec{L}$ does not change, while the direction of $\vec{L}$ is changing. Because the change in angular momentum $d\vec{L}$ is in the direction of $\vec{M}$, which lies in the xy plane, the gyroscope undergoes precessional motion.

As one can see from the Figure 1.24 (inset), the magnitude $dL \approx Ld\varphi$, where $d\varphi$ is the angle through which the gyroscope axis rotates. Then, $d\varphi = \frac{dL}{L}$, and the angular speed of precession $\omega_{pr} = \frac{d\varphi}{dt} = \frac{1}{L}\frac{dL}{dt} = \frac{1}{L}M_O = \frac{mgR_C}{I\omega}$, where $M_O = mgR_C$ is the gravitational torque about the point O, R_C is radius of the circle traced out by the center of mass of the gyroscope.

Thus, **precessional frequency** $\omega_{pr} = \frac{mgR_C}{I\omega}$. (1.84)

The axis of symmetry is one of the principal axes of inertia of the gyroscope. Rotation about this axis is steady, the angular momentum of the gyroscope coincides with the direction of the axis of rotation and the axis of rotation keeps its direction unchanged in the case of a shift. In order to change the direction of the gyroscope axis relative to the fixed reference frame the external torque should be exerted. Such phenomenon is called a gyroscopic effect.

Questions for self-control

1. Rolling motion of the rigid body
2. Equations of dynamics describing the rolling motion
3. Total kinetic energy of the rolling motion
4. Inertia tensor
5. Principal axes of inertia.
6. Gyroscope motion
7. Precession frequency

Topic 1.5. Statics and dynamics of liquids and gases

Lecture 6

1.5.1 Fluid statics

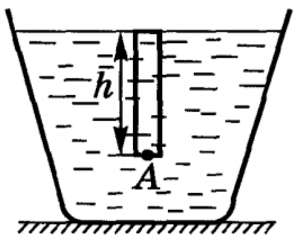
A **fluid** is a collection of molecules that are randomly arranged and held together by weak cohesive forces and by forces exerted by the walls of a container. Both liquids and gases are fluids.

Fluid statics is the mechanics of incompressible fluids at rest. Incompressible fluid is the fluid whose density is uniform throughout the liquid.

Consider an object immersed into the fluid. The only stress that can be exerted on an object submerged in a static fluid is one that tends to compress the object from all sides. In other words, the force exerted by a static fluid on an object is always perpendicular to the surfaces of the object.

Pressure is the force applied perpendicular to the surface of an object per unit area over which that force is distributed:

$$P = \frac{F}{S} \tag{1.85}$$



Hydrostatic pressure is the pressure of incompressible fluids at rest due to the weight of the fluid. In a conservative force field like a gravitational force field, pressure exerted by a fluid at equilibrium becomes a function of force exerted by gravity:

$P = \frac{F_{gr}}{S} = \frac{mg}{S} = \frac{\rho Shg}{S} = \rho hg$, where ρ is the fluid density, h is the height of fluid column, g is the gravitational acceleration.

$$P = \rho gh\,. \tag{1.86}$$

Pressure at a depth h below the surface of the liquid is the sum of the atmospheric pressure P_0 (if the liquid is open to the atmosphere) and ρgh:

$$P = P_0 + \rho g h . \tag{1.87}$$

Because the pressure in a fluid depends on depth and on the value of P_0, any increase in pressure at the surface must be transmitted to every other point in the fluid.

Pascal's Law: The external pressure exerted anywhere in a confined fluid is transmitted equally to all points in all directions throughout the fluid.

Equivalently, the pressure acting on the liquid or gas is passed without changes to any part of the liquid or gas.

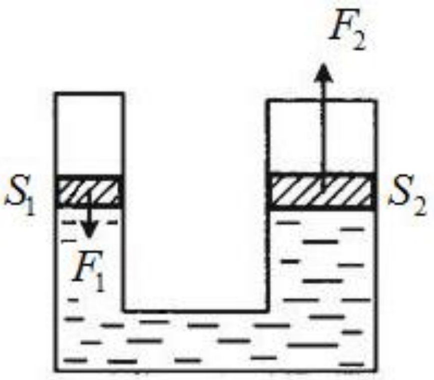


Figure 1.25.

An important application of Pascal’s law is the **hydraulic press** illustrated in Figure 1.25. A force of magnitude F_1 is applied to a small piston of surface area S_1. The pressure is transmitted through an incompressible liquid to a larger piston of surface area S_2. Because the pressure must be the same on both sides,

$$P_1 = P_2 ; \quad \frac{F_1}{F_2} = \frac{S_1}{S_2} .$$

By designing a hydraulic press with appropriate areas S_1.and S_2, a large output force can be applied by means of a small input force.

Law of communicating vessels: a homogeneous liquid balances out to the same level in all of the communicating containers regardless of the shape and volume of the containers.

If the communicating containers are filled with different unmixable liquids, then according to the Pascal’s law, $P_1 = P_2$, so $P_{01} + \rho_1 g h_1 = P_{02} + \rho_2 g h_2$. If the vessels are open, $P_{01} = P_{02} = P_0$, and $\rho_1 g h_1 = \rho_2 g h_2$.

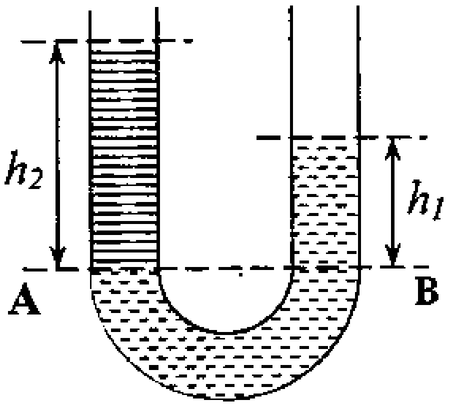


Figure 1.26.

Levels of the unmixable liquids in communicating vessels:

$$\frac{h_1}{h_2} = \frac{\rho_2}{\rho_1}, \qquad (1.88)$$

where h_1, h_2 are heights of the liquid columns; ρ_1, ρ_2 are the densities of the liquids (Figure 1.26).

If the liquid is homogeneous, $\rho_1 = \rho_2 = \rho$, and $h_1 = h_2$.

<u>A buoyant force</u> is the upward force exerted by a fluid on any immersed object.

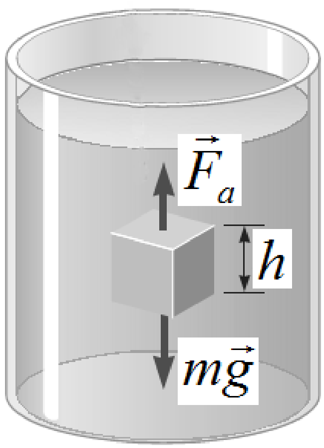


Figure 1.27.

<u>Archimedes's principle</u>: the magnitude of the buoyant force on an object immersed into the fluid is equal to the weight of the fluid displaced by the object or by the immersed part of the object.

$$F_a = \rho_f g V_b, \qquad (1.89)$$

where ρ_f is the density of the fluid; V_b is the volume of the immersed body, or the volume of the displaced fluid.

To better understand the origin of the buoyant force, consider a cube of solid material immersed in a liquid as in Figure 1.27. The pressure P_{bot} at the bottom of the cube is greater than the pressure P_{top} at the top by an amount $\rho_f gh$, where h is the height of the cube and ρ_f is the density of the fluid. Then, a force exerted on the cube by the fluid will be

$$F = (P_{bot} - P_{top})S = (\rho_f g h_{bot} - \rho_f g h_{top})S = \rho_f g h S = \rho_f g V_b.$$

Case 1: Totally Submerged Object

When an object is totally submerged in a fluid of density ρ_f, the volume V_{disp} of the displaced fluid is equal to the volume V_{obj} of the object; so, the magnitude of the upward buoyant force is $F_a = \rho_f g V_{obj}$. If the object has a mass m and density ρ_{obj}, the net force on the object will be a difference of the gravitational force and the buoyant force: $F_{net} = F_a - mg = \rho_f gV_{obj} - \rho_{obj} gV_{obj} = (\rho_f - \rho_{obj})gV_{obj}$.

Hence, if the density of the object is less than the density of the fluid, the downward gravitational force is less than the buoyant force and the unsupported object accelerates upward. If the density of the object is greater than the density of the fluid, the upward buoyant force is less than the downward gravitational force and the unsupported object sinks. If the density of the submerged object equals the density of the fluid, the net force on the object is zero and the object remains in equilibrium. Therefore, the direction of motion of an object submerged in a fluid is determined only by the densities of the object and the fluid.

Case 2: Floating Object

Now consider an object of volume V_{obj} and density $\rho_{obj} < \rho_f$ in static equilibrium floating on the surface of a fluid, that is, an object that is only partially submerged. In this case, the upward buoyant force is balanced by the downward gravitational force acting on the object. If V_{disp} is the volume of the fluid displaced by the object (this volume is the same as the volume of that part of the object beneath the surface of the fluid), the buoyant force has a magnitude $F_a = \rho_f gV_{disp}$. For the equilibrium condition $F_{net} = 0$, and $F_a = mg$; $\rho_f gV_{disp} = \rho_{obj} gV_{obj}$. Or,

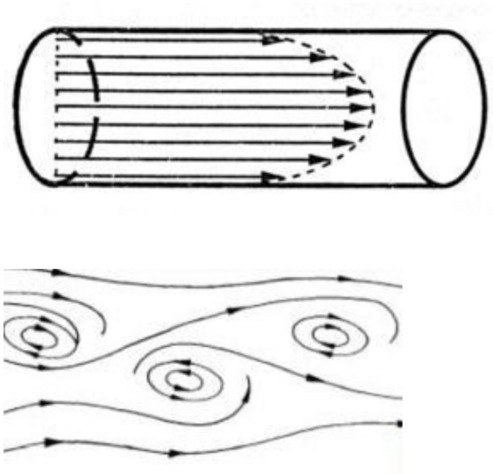
$$\frac{\rho_f}{\rho_{obj}} = \frac{V_{obj}}{V_{disp}}.$$

This equation shows that the fraction of the volume of a floating object that is below the fluid surface is equal to the ratio of the density of the object to that of the fluid.

1.5.2 Fluid dynamics

Fluid dynamics is the mechanics of incompressible fluids in motion.

The flow is **laminar** if each particle of the fluid follows a smooth path such that the paths of different particles never cross each other as shown in Figure 1.28 (a). A fluid flows in parallel layers, with no disruption between the layers. There is no lateral mixing, and adjacent layers slide past one another. In **steady flow**, every fluid particle arriving at a given point has the same velocity.



(a) *(b)*

Figure 1.28: (a) laminar flow, (b) turbulent flow

Above a certain critical speed, fluid flow becomes turbulent. **Turbulent flow** is irregular flow characterized by small whirlpool-like regions as shown in Figure 1.28 (b).

Viscosity is characteristic of the degree of internal friction in the fluid. This **internal friction**, or viscous force, is associated with the resistance that two adjacent layers of fluid have to moving relative to each other.

It is a property of the fluid which opposes the relative motion between the two surfaces of the fluid in a fluid that are moving at different velocities. Viscosity causes part of the fluid's kinetic energy to be converted to internal energy.

Because the motion of real fluids is very complex and not fully understood, we make some simplifying assumptions. Let's consider the **ideal fluid flow** making the following four assumptions:

1. The fluid is nonviscous. In a nonviscous fluid, internal friction is neglected. An object moving through the fluid experiences no viscous force.

2. The flow is steady. In steady (laminar) flow, all particles passing through a point have the same velocity.

3. The fluid is incompressible. The density of an incompressible fluid is constant.

4. The flow is irrotational. In irrotational flow, the fluid has no angular momentum about any point. If a small paddle wheel placed anywhere in the fluid does not rotate about the wheel's center of mass, the flow is irrotational.

The flow lines are the visualization of the fluid flow in a region of space:

– The velocity vector is tangent to the flow field line at each point.

– The number of lines per unit area is proportional to the magnitude of the flow velocity in that region.

For the ideal fluid flow:

1) **Equation of continuity**: the product of the area and the fluid speed at all points along a pipe is constant for an incompressible fluid:

$vS = const.$ (1.90)

Or, $v_1 S_1 = v_2 S_2$, where v_1, v_2 is the fluid speed at the cross-section of the pipe of area S_1, S_2 (see Figure 1.29).

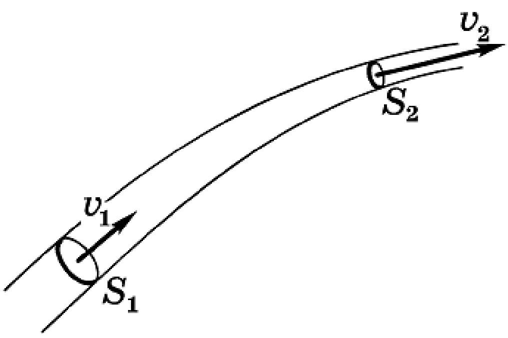


Figure 1.29.

This equation is a sequence from the mass conservation law: because the fluid is incompressible and the flow is steady, the mass of fluid that passes point 1 in a time interval dt must equal the mass that passes point 2 in the same time interval. That is,

$$m_1 = m_2; \qquad \rho_1 S_1 dl_1 = \rho_2 S_2 dl_2; \qquad \rho_1 S_1 v_1 dt = \rho_2 S_2 v_2 dt,$$

where $dl = vdt$ is the distance covered by the fluid in the time interval dt. Density of the ideal fluid is unchanging, $\rho_1 = \rho_2 = \rho$, and we obtain $S_1 v_1 = S_2 v_2$.

2) **Bernoulli's principle**: for the steady flow of ideal incompressible fluid the total pressure equal to the sum of static pressure (P), pressure due to weight (ρgh) and dynamic pressure ($\frac{\rho v^2}{2}$), is constant along the flow.

$$P_1 + \rho g h_1 + \frac{\rho v_1^2}{2} = P_2 + \rho g h_2 + \frac{\rho v_2^2}{2} = const, \qquad (1.91)$$

where h_1, h_2 is the height of elevation level, ρ is the fluid density, P_1, P_2 is the static pressure at the chosen cross-section of the pipe, v_1, v_2 is the fluid flow speed at the chosen cross-section of the pipe.

An increase in the speed of a fluid occurs simultaneously with a decrease in pressure, and the pressure decreases as the elevation increases.

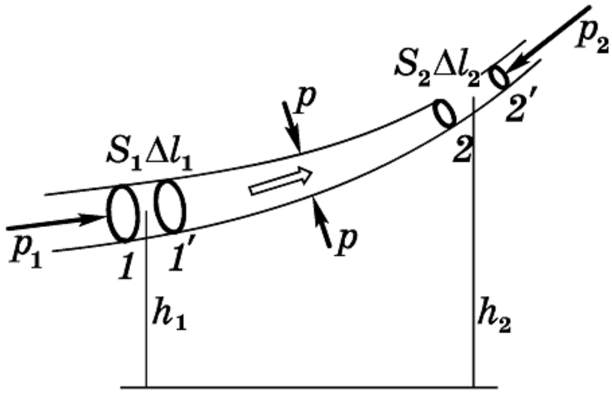


Figure 1.30.

As a fluid moves through a region where its speed or elevation above the Earth's surface changes, the pressure in the fluid varies with these changes. Let's consider a segment of fluid between the cross-sections 1 and 2 (Figure 1.30). In a time interval *dt* this segment will be between the cross-sections 1' and 2'.

As we know, change in the total mechanical energy of the segment in time *dt* equals to the work by external forces on this segment:

$\Delta E = A_{ex}$.

Here external forces are the forces of pressure exerted on the fluid segment, and only the forces of pressure perpendicular to cross-sections of the segment do work on it.

$F_1 = P_1 S_1,\ F_2 = P_2 S_2$.

$A_1 = F_1 \Delta l_1 = P_1 S_1 \Delta l_1 = P_1 \Delta V_1$ and, similarly, $A_2 = -P_2 \Delta V_2$. A_2 is negative because the force on the segment of fluid is to the left and the displacement of the point of application of the force is to the right. The volumes $\Delta V_1, \Delta V_2$ are equal because the fluid is incompressible. Then, the work by the external forces

$A_{ex} = (P_1 - P_2)\Delta V$.

Part of this work goes into changing the kinetic energy of the segment of fluid, and part goes into changing the gravitational potential energy of the segment–Earth system.

$$\Delta\text{E}=\Delta(K+U) = \left(\frac{\rho \Delta V v_2^2}{2} + \rho \Delta V g h_2\right) - \left(\frac{\rho \Delta V v_1^2}{2} + \rho \Delta V g h_1\right),$$

where $\rho\Delta V$ is the mass of the segment of fluid, ρ is the fluid density.

After substituting and dividing each term by the portion volume ΔV, we obtain

$$P_1 - P_2 = \frac{\rho v_2^2}{2} - \frac{\rho v_1^2}{2} + \rho g h_2 - \rho g h_1, \text{ or } P_1 + \rho g h_1 + \frac{\rho v_1^2}{2} = P_2 + \rho g h_2 + \frac{\rho v_2^2}{2}.$$

1.5.3 Applications of fluid dynamics

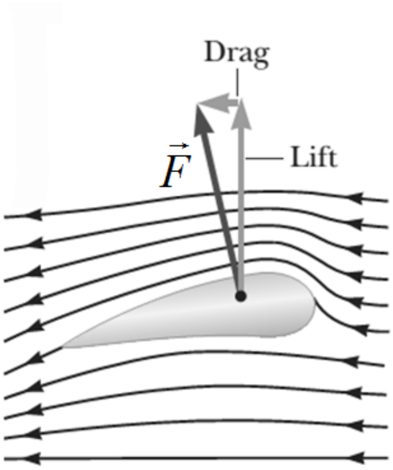


Figure 1.31.

Consider the streamlines that flow around an airplane wing as shown in Figure 1.31. Let's assume the airstream approaches the wing horizontally from the right with a velocity $\vec{v}_1$. The tilt of the wing causes the airstream to be deflected downward with a velocity $\vec{v}_2$. Because the airstream is deflected by the wing, the wing must exert a force on the airstream.

According to Newton's third law, the airstream exerts a force $\vec{F}$ on the wing that is equal in magnitude and opposite in direction. This force has a vertical component called lift (or **aerodynamic lift**) and a horizontal component called **drag**. The lift depends on several factors, such as the speed of the airplane, the area of the wing, the wing's curvature, and the angle between the wing and the horizontal. The curvature of the wing surfaces causes the pressure above the wing to be lower than that below the wing due to the Bernoulli effect. This pressure difference assists with the lift on the wing.

Questions for self-control

1. Definition of the fluid
2. What is a hydrostatic pressure?
3. Pascal's law. Law of communicating vessels
4. Archimedes' principle. Conditions for flotation of a body
5. Assumptions for the ideal fluid flow. Laminar and turbulent flow
6. Equation of continuity
7. Bernoulli's principle
8. What is the aerodynamic lift?

Topic 1.6. Oscillatory motion

Lecture 7

1.6.1 Simple harmonic motion

Oscillatory motion is motion of an object that regularly returns to a given position after a fixed time interval called the period of oscillations.

Simple harmonic motion is a type of oscillation motion when position changes with time according to the harmonic law (Figure 1.32):

$$x(t) = A\cos(\omega_0 t + \varphi_0), \tag{1.92}$$

where A is the **amplitude** of the motion – the maximum displacement from the equilibrium position;

$\varphi = (\omega_0 t + \varphi_0)$ is the **phase** of the oscillation, φ_0 is the **phase constant** (or initial phase angle) determined by the position of the particle at t = 0;

ω_0 is the **angular frequency** of oscillations.

Period T of the motion is the time interval required for the particle to go through one full cycle of its motion:

$$T = \frac{1}{\upsilon} = \frac{2\pi}{\omega}.$$

Frequency υ (Hz) is the number of oscillations the particle undergoes per unit time interval.

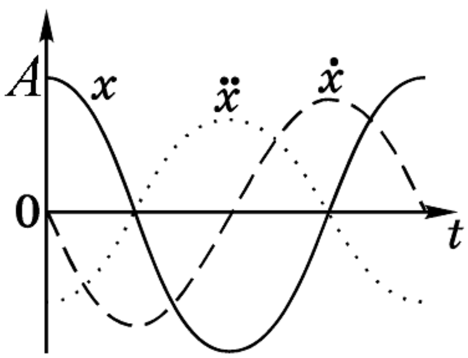


Figure 1.32 Position x, velocity $\dot{x} = \frac{dx}{dt}$ and acceleration $\ddot{x} = \frac{d^2x}{dt^2}$ of a particle in the simple harmonic motion; A is the amplitude.

Velocity of a particle in simple harmonic motion:

$$v = \frac{dx(t)}{dt} = -v_{\max}\sin(\omega_0 t + \varphi_0) = v_{\max}\cos(\omega_0 t + \varphi_0 + \frac{\pi}{2}), \tag{1.93}$$

where $v_{\max} = A\omega_0$ is the maximum magnitude of velocity (amplitude of velocity)

Acceleration of a particle in simple harmonic motion:

$$a = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} = -A\omega_0^2\cos(\omega_0 t + \varphi_0) = a_{\max}\cos(\omega_0 t + \varphi_0 + \pi), \tag{1.94}$$

where $a_{\max} = A\omega_0^2$ is the maximum magnitude of acceleration (amplitude of acceleration).

As we see, velocity and acceleration of the particle are also changing with time following the harmonic law, what is more, velocity has phase advance π/2 and acceleration has phase advance π with respect to the displacement *x*. Displacement and acceleration are in the antiphase. Also, we can see that $\frac{d^2x}{dt^2} = -\omega_0^2 x$, or

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0. \tag{1.95}$$

This differential equation is the **equation of harmonic oscillator**.

If equation of motion of a particle can be transformed into the equation of harmonic oscillator, then we can state that this particle is under simple harmonic motion with angular frequency ω_0 equal to the square root of multiplier of *x*.

The net force causing simple harmonic motion of the particle of mass *m* has a form of

$$F = ma = m\frac{d^2x}{dt^2} = -m\omega_0^2 x = -kx, \tag{1.96}$$

where $k = m\omega_0^2$ is constant.

F is called a **restoring force** because it is always directed toward the equilibrium position (for which $x = 0$) and therefore opposite the displacement of the particle from equilibrium.

We see that *simple harmonic motion is a type of oscillation motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.*

The restoring force must be elastic or quasi-elastic. Quasi-elastic force is the force not elastic in its nature, but subject to the law

$$F = -kx.$$

So, the particle moves with simple harmonic motion whenever its acceleration given by the net force is proportional to its position and is oppositely directed to the displacement from equilibrium.

1.6.2 Spring pendulum

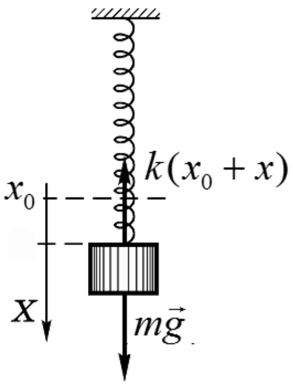


Figure 1.33.

Spring pendulum is a block of mass *m* attached to the end of a spring with stiffness constant *k* (Figure 1.33).

Consider a block hanging from the spring. According to the Newton's second law, $ma = mg - F_{el} = mg - k(x_0 + x)$, where x_0 is the equilibrium position: $mg = kx_0$. Then, $m\dfrac{d^2x}{dt^2} = -kx$, and

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \qquad (1.97)$$

We have obtained the equation of harmonic oscillator, and the angular frequency equals

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

Period of the spring pendulum is $T = 2\pi\sqrt{\frac{m}{k}}$.

1.6.3 Mathematical pendulum

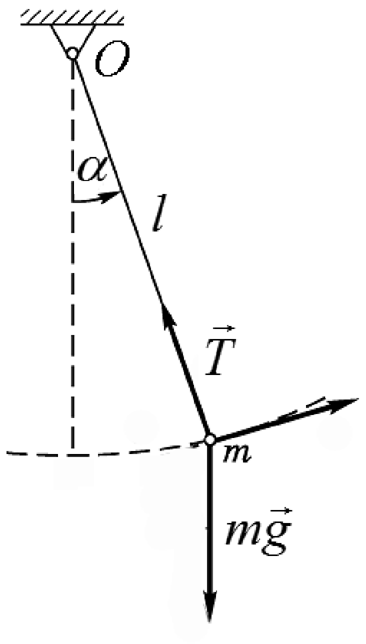


Figure 1.34.

Mathematical pendulum (**simple gravity pendulum**) is a particle-like bob of mass *m* suspended by a massless, inextensible string of length *l* and exhibiting oscillatory motion in the vertical plane (Figure 1.34).

The length of the pendulum *l* must be much larger than the size of the bob so it can be considered as a point mass.

It is more convenient to consider the gravity pendulum in terms of the rotational dynamics. Considering α as the angular displacement of the bob, let us write the fundamental equation of dynamics of rotational motion of the bob about the point *O*:

$$I\beta = M,$$

where *I* is the moment of inertia, for the single point mass $I = ml^2$; β is the angular acceleration, $\beta = \frac{d^2\alpha}{dt^2}$; *M* is the net torque on the bob. We can see that the only non-zero torque about the point *O* is the torque due to the gravitational force, which equals $M = -mgl\sin\alpha$. By substituting these quantities into the fundamental equation of dynamics of rotational motion, we obtain

$$l\frac{d^2\alpha}{dt^2} = -g\sin\alpha.$$

This expression is not of the same mathematical form the equation of harmonic oscillator. However, if we assume α is small (less than about 10^0 or 0.2 rad), we can use the **small angle approximation**, in which $\sin\alpha \approx \alpha$, where α is measured in radians.

Therefore, for small angles, the equation of motion becomes $l\frac{d^2\alpha}{dt^2} = -g\alpha$;

$$\frac{d^2\alpha}{dt^2} + \frac{g}{l}\alpha = 0. \quad (1.98)$$

This is the equation of harmonic oscillator with the angular frequency

$$\omega_0 = \sqrt{\frac{g}{l}}.$$

Period of the simple gravity pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$.

If the point of suspension of the simple pendulum moves with an acceleration $\vec{a}$ in the inertial reference frame, then the period:

$$T = 2\pi\sqrt{\frac{l}{g_l}}, \qquad \text{where} \quad g_l = |\vec{g} - \vec{a}| \quad (1.99)$$

1.6.4 Physical pendulum

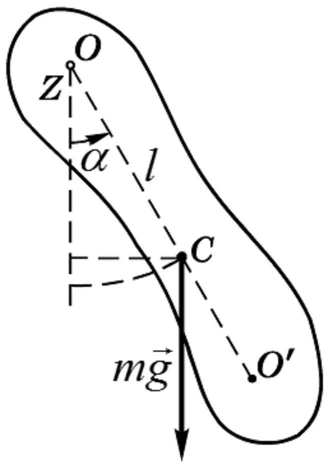


Figure 1.35.

<u>Physical pendulum</u> is a rigid body of mass *m* suspended at the fixed point *O* and exhibiting oscillatory motion about that point under the action of gravity. We can say that the physical pendulum undergoes fixed axis rotation about a fixed point *O*.

Similarly to the simple gravity pendulum, it is convenient to consider the motion of physical pendulum in terms of the rotational dynamics approach. Considering α as the angular displacement of the center of mass of the body, let us write the fundamental equation of dynamics of rotational motion of the body about the point *O*:

$$I\beta = M,$$

where I is the moment of inertia of the given body about the point O; β is the angular acceleration, $\beta = \dfrac{d^2\alpha}{dt^2}$; M is the net torque on the body. We can see that the only non-zero torque about the point O is the torque due to the gravitational force, which equals $M = -mgl\sin\alpha$, where l is the distance from the point O to the center of mass C of the body. By substituting these quantities into the fundamental equation of dynamics of rotational motion and applying the small angle approximation, $\sin\alpha \approx \alpha$, we obtain the equation of motion:

$$\frac{d^2\alpha}{dt^2} + \frac{mgl}{I}\cdot\alpha = 0. \qquad (1.100)$$

This is the equation of harmonic oscillator with the angular frequency

$$\omega_0 = \sqrt{\frac{mgl}{I}}.$$

Period of the physical pendulum is $T = 2\pi\sqrt{\dfrac{I}{mgl}}$.

1.6.5 Total energy of a simple harmonic oscillator

$$\begin{aligned} E &= \mathrm{K} + U = \frac{mv^2}{2} + \frac{kx^2}{2} = \frac{1}{2}\Big(m\big(-A\omega_0\sin(\omega_0 t + \varphi_0)\big)^2 + k(A\cos(\omega_0 t + \varphi_0))^2\Big) = \\ &= \frac{A^2}{2}\Big(m\omega_0^2\sin^2(\omega_0 t + \varphi_0) + k\cos^2(\omega_0 t + \varphi_0)\Big) = \frac{m\omega_0^2 A^2}{2} = \frac{kA^2}{2} \end{aligned} \qquad (1.101)$$

Kinetic and potential energy of the particle in simple harmonic motion are phase-shifted by $\pi/2$ with respect to each other: when kinetic energy is maximal, the potential energy is minimal, but the total mechanical energy is conserved.

The mean over period magnitude of energy $\langle K\rangle = \langle U\rangle = E/2$.

1.6.6 Addition of oscillations

1.6.6.1. Addition of oscillations in one direction

Let's consider addition of two oscillations of the same frequency $\omega_1 = \omega_2 = \omega$ but at different amplitudes and phase angles. The resulting oscillation will be

$x = x_1 + x_2 = A_1 \cos(\omega t + \varphi_1) + A_2 \cos(\omega t + \varphi_2)$.

It will also be a harmonic oscillation with frequency *ω*:

$x = A\cos(\omega t + \varphi)$

Let's find *A* and *φ*. Each of the oscillations can be depicted as a vector $\vec{A}_i$ of magnitude A_i directed at the angle φ_i, and the vector diagram can be built (Figure 1.36):

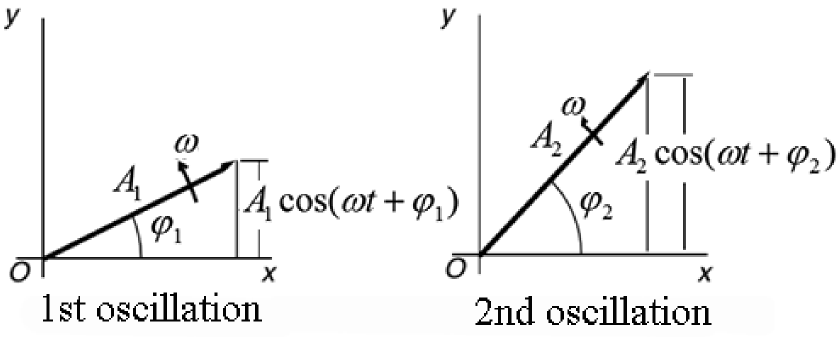


Figure 1.36.

The resulting oscillation will be the sum of these two vectors $\vec{A} = \vec{A}_1 + \vec{A}_2$ (Figure 1.37):

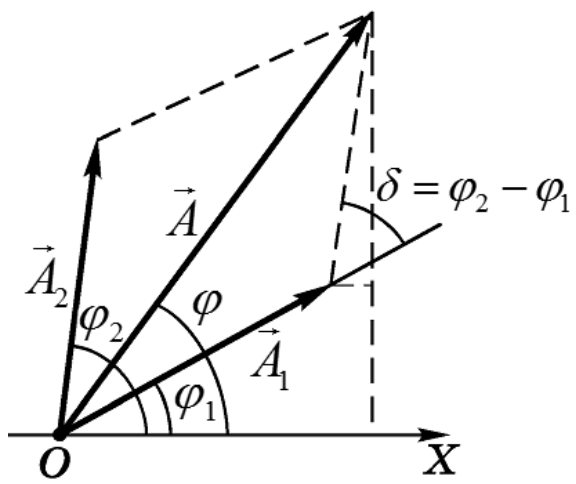


Figure 1.37.

According to the parallelogram law of vector addition,

$$A = A_1^2 + A_2^2 + 2A_1 A_2 \cos\delta ,$$

$$\varphi = \frac{A_1 \sin\varphi_1 + A_2 \sin\varphi_2}{A_1 \cos\varphi_1 + A_2 \cos\varphi_2}, \qquad (1.102)$$

where $\delta = \varphi_2 - \varphi_1$ is the phase difference of these two oscillations.

1.6.6.2 Beat of oscillations

Beat is the result of addition of two oscillations of slightly different frequencies, $|\omega_1 - \omega_2| << \omega_1 \ or \ \omega_2$, which can be considered as a harmonic oscillation whose amplitude periodically varies with time at rate equal to the difference of the two frequencies $|\omega_1 - \omega_2|$ (Figure 1.38).

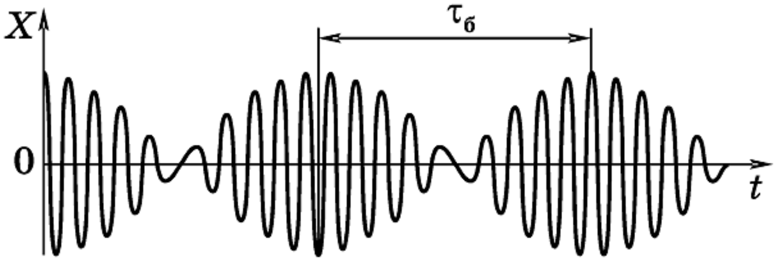


Figure 1.38.

The beat period τ_b can be found from the condition $|\omega_1 - \omega_2|\tau_b = 2\pi$.

Amplitude of the resulting oscillation can be found from the vector diagram similarly to the previous case, but we must take into account that phase difference of the two oscillations depends on time now: $\delta = \varphi_2 - \varphi_1 + (\omega_2 - \omega_1)t$.

1.6.6.3 Addition of perpendicular oscillations

In the simplest case of equal frequencies $\omega_1 = \omega_2 = \omega$, equations of these oscillations

$$x = A\cos(\omega t), \qquad y = B\cos(\omega t + \delta).$$

If the phase difference $\delta = 0$ or $\delta = \pi$ the result of addition will be a straight line $y = \pm\frac{B}{A}x$. That is, the resulting trajectory of the oscillating particle will be a straight line.

If the phase difference $\delta = \pi/2$ the result of addition will be an ellipse: $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$

In general case, when the two perpendicular oscillations have different frequencies and phases, the result of addition will be a complicated trajectory called the **<u>Lissajous curve</u>.**

1.6.7 Damped Oscillations

The oscillatory motions we have considered so far have been for ideal systems, that is, systems that oscillate indefinitely under the action of only one force, a linear restoring force. In many real systems, nonconservative forces such as friction or air resistance retard the motion. Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be damped.

If a frictional force (damping) proportional to the velocity is also present, the harmonic oscillator is called a **damped oscillator**. Depending on the friction coefficient, the system can:

- Oscillate with a frequency lower than in the non-damped case, and an amplitude decreasing with time (*underdamped* oscillator).
- Decay to the equilibrium position, without oscillations (*overdamped* oscillator).

The retarding force is proportional to the speed of the moving object and acts in the direction opposite the velocity of the object with respect to the medium:

$\vec{F}_{ret} = -b\vec{v}$, where b is a constant called the damping coefficient. We can write Newton's second law as

$$ma_x = -bv_x - kx; \qquad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0; \qquad \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0.$$

The solution of this differential equation is

$$x = Ae^{-(b/2m)t}\cos(\omega t + \varphi_0), \qquad (1.103)$$

where $A_{dam} = Ae^{-(b/2m)t}$ is the exponentially decreasing amplitude of damped oscillations (see Figure 1.39);

$\omega = \sqrt{\left(\frac{k}{m}\right)^2 - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$ is the angular frequency of damped oscillation;

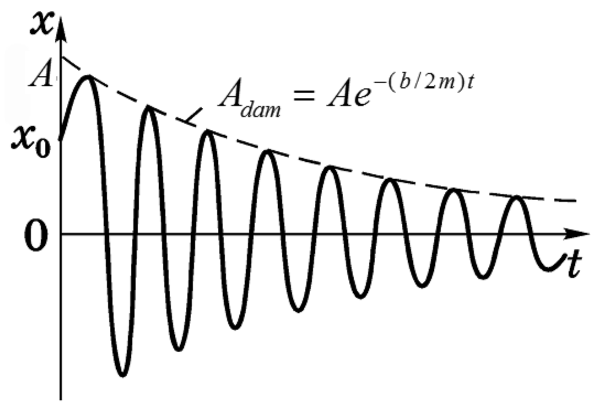


Figure 1.39.

$\omega_0 = \sqrt{k/m}$ represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the **natural frequency** of the system.

When the magnitude of the retarding force is small such that $b/2m < \omega_0$, the system is underdamped. If the medium is so viscous that the retarding force is large compared with the restoring force $b/2m > \omega_0$, the system is overdamped and does not oscillate.

1.6.8 Forced Oscillations

The mechanical energy of a damped oscillator decreases in time as a result of the resistive force. It is possible to compensate for this energy decrease by applying a periodic external force that does positive work on the system. At any instant, energy can be transferred into the system by an applied force that acts in the direction of motion of the oscillator. The amplitude of motion remains constant if the energy input per cycle of motion exactly equals the decrease in mechanical energy in each cycle that results from resistive forces.

A forced oscillator is a damped oscillator driven by an external force that varies periodically,

$F(t) = F_0 \sin \omega t$, where F_0 is a constant and ω is the angular frequency of the driving force. In general, the frequency of the driving force is variable, whereas the natural frequency ω_0 of the oscillator is fixed.

Newton's second law in this situation gives

$$ma_x = F_0 \sin \omega t - bv_x - kx\,; \qquad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin \omega t\,. \qquad (1.104)$$

The solution of this equation is rather complicated. After a sufficiently long period of time, when the energy input per cycle from the driving force equals the amount of mechanical energy transformed to internal energy for each cycle, a steady-state condition is reached in which the oscillations proceed with constant amplitude. In this situation, the solution of the equation will be

$$x = A\cos(\omega t + \varphi), \qquad \text{where } A = \frac{F_0/m}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\dfrac{b\omega}{m}\right)^2}}. \tag{1.105}$$

The forced oscillator vibrates at the frequency of the driving force.

When the frequency of the driving force is near the natural frequency of oscillation, $\omega \approx \omega_0$, the dramatic increase in amplitude occurs. This phenomenon is called **resonance**. The **resonance frequency** of the system can be found as the frequency when the $A(\omega)$ dependence reaches its maximum (Figure 1.40). Using the condition for the function maxima, $dA/d\omega = 0$, we obtain

$$\omega_{res} = \sqrt{\omega_0^2 - 2\left(\frac{b}{2m}\right)^2}\,; \qquad A_{res} = \frac{F_0}{b\sqrt{\omega_0^2 - \left(\dfrac{b}{2m}\right)^2}}. \tag{1.106}$$

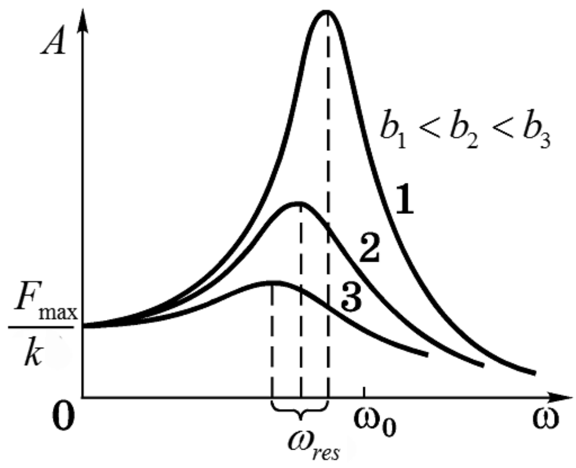


Figure 1.40.

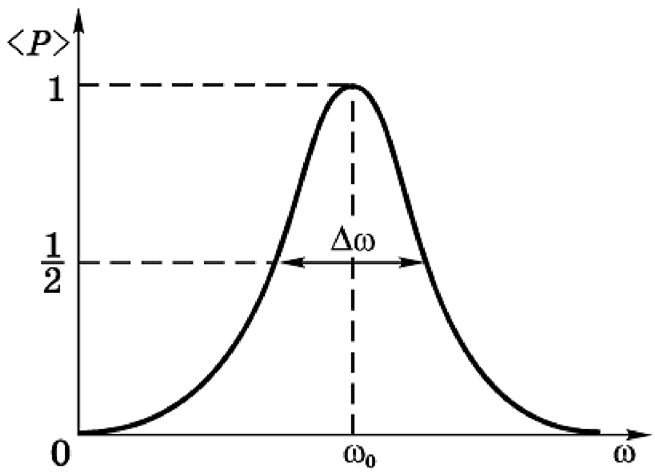


Figure 1.41.

The **quality factor** or ***Q* factor** is a dimensionless parameter that characterizes a resonator's bandwidth relative to its center frequency. Higher Q indicates a lower rate of energy loss relative to the stored energy of the resonator; the oscillations die out more slowly. Resonators with high quality factors have low damping so that they ring or vibrate longer.

$$Q = \frac{\omega_0}{\Delta\omega}, \tag{1.107}$$

where ω_0 is the resonance frequency, $\Delta\omega$ the resonance width, i.e. the bandwidth over which the power of vibration is greater than half the power at the resonant frequency (the width of the resonance peak measured at the height of $\frac{1}{\sqrt{2}}$ times maximum amplitude on the amplitude vs frequency plot; or at the height of $\frac{1}{2}$ maximum amplitude on the power vs frequency plot, see Figure 1.41).

The other common equivalent definition for Q is the ratio of the energy stored in the oscillating resonator to the energy dissipated per cycle by damping processes:

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy dissipated per cycle}}.$$

Questions for self-control

1. What is a simple harmonic motion?
2. Characteristics of the harmonic oscillations
3. What is the difference between mathematical and physical pendulums?
4. Equation of the harmonic oscillator
5. What is the significance of the small angle approximation?
6. Energy of the harmonic oscillator
7. The diagram method of addition of oscillations in one direction.
8. What is a beat of oscillations?
9. Damped oscillations.
10. Forced oscillations.
11. Resonance.
12. Q factor.

Topic 1.7. Elastic waves

Lecture 8

1.7.1 Elastic waves

Wave is a periodic disturbance propagating through a medium. Wave motion transfers energy through space without the accompanying transfer of matter.

Mechanical wave is the wave that is an oscillation of matter, and therefore transfers energy through a medium.

All mechanical waves require (1) some source of disturbance, (2) a medium containing elements that can be disturbed, and (3) some physical mechanism through which elements of the medium can influence each other.

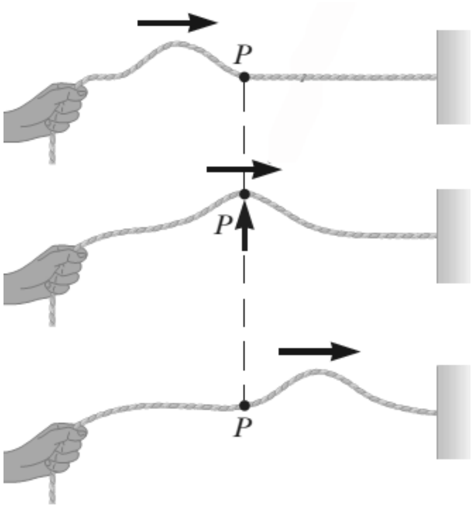


Figure 1.42.

For example, consider a long string that is under tension and has its one end fixed (Figure 1.42). If we flick the free end of the string, a single bump (called a pulse) is formed and travels along the string with a definite speed. The hand is the source of the disturbance. The string is the medium through which the pulse travels—individual elements of the string are disturbed from their equilibrium position.

The string itself does not propagate, but the pulse propagates at definite speed.

Mechanical waves are elastic. Waves propagate through the medium, and the substance of this medium is deformed. The deformation of the medium generates the restoring forces, so the oscillations of the medium are around almost fixed locations while the energy of the oscillation propagates trough space.

Transverse wave is a traveling wave or pulse that causes the elements of the disturbed medium to move perpendicular to the direction of propagation (Figure 1.43).

– Transverse mechanical waves can only move through solid materials.

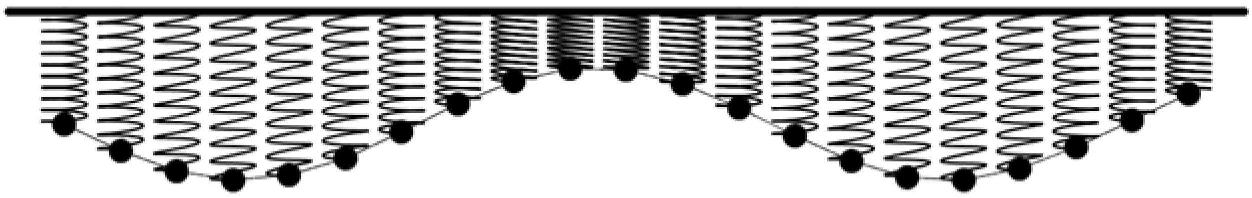


Figure 1.43.

Longitudinal wave is a traveling wave or pulse that causes the elements of the medium to move parallel to the direction of propagation (Figure 1.44). Propagation of the longitudinal waves creates region of compression (an area of high molecular density and pressure) and rarefaction (an area of low molecular density and pressure) in the medium.

– Longitudinal waves can move through solids, liquids, and gases.

Figure 1.44.

Wavelength λ is the spatial period of the wave – the minimum distance between any two identical points along the direction of the wave propagation, the phase difference of which is 2π radians.

The oscillating points are in antiphase if the distance between them is $\lambda/2$.

The wavelength is the distance of the wave propagation over the time interval equal to the period of oscillations.

$$\lambda = \upsilon T = \frac{v}{\upsilon} = \frac{2\pi v}{\omega}, \tag{1.108}$$

where T is the period; υ is the frequency; ω is the angular frequency; v is the speed of the wave propagation.

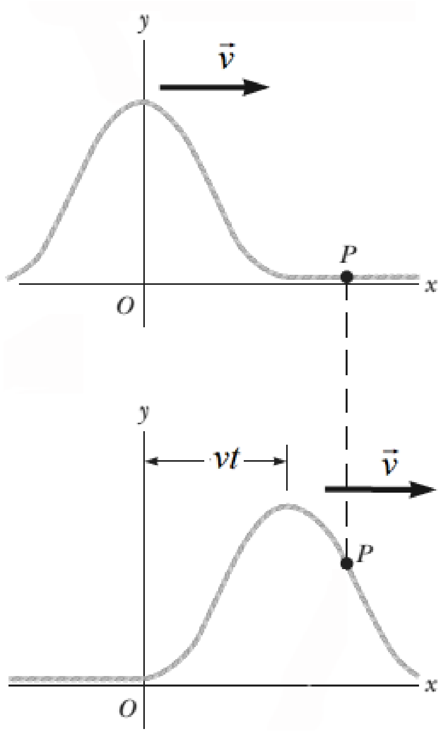


Figure 1.45.

Consider a pulse traveling to the right on a long string as shown in Figure 1.45. At time $t = 0$ the shape of the pulse can be represented by some mathematical function $y(x, 0) = f(x)$, which describes the transverse position y of the element of the string located at each value of x at time $t = 0$. Let the speed of the pulse be v, then the pulse travels to the right a distance vt at the time t (Figure 1.45). We assume the shape of the pulse does not change with time. Therefore, at time t, the shape of the pulse is the same as it was at time $t = 0$. Consequently, an element of the string at x at this time has the same y position as an element located at $x - vt$ had at time $t = 0$.

This way we can represent the transverse position y for all positions and times as $y(x,t) = f(x - vt)$. Similarly, if the pulse travels to the left, the transverse positions of elements of the string are described by $y(x,t) = f(x + vt)$. The function y is called the **wave function** and depends on the two variables x and t.

Sinusoidal wave is the wave whose wave function is described by the sine law:

$$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right), \tag{1.109}$$

where A is the amplitude of the wave, λ is the wavelength.

By definition, the wave travels through a displacement Δx equal to one wavelength λ in a time interval Δt of one period T. Therefore, the wave speed, wavelength, and period are related by the expression

$$v = \frac{\Delta x}{\Delta t} = \frac{\lambda}{T}.$$

Substituting this expression for v into the wave equation gives

$$y(x,t) = A\sin\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right). \tag{1.110}$$

We can express the wave function in a convenient form by defining two other quantities, the angular wave number k (usually called simply the **wave number**) and the angular frequency ω:

$$k = \frac{2\pi}{\lambda}; \qquad \omega = \frac{2\pi}{T}.$$

Using these definitions, the wave equation takes a more compact, general form:

$$y = A\sin\left(kx - \omega t + \varphi_0\right), \tag{1.111}$$

where φ_0 is the phase constant given by the initial conditions, $\varphi = kx - \omega t + \varphi_0$ is the phase of the wave.

A **wavefront** is the locus of points characterized by propagation of position of the same phase. The **phase velocity** of a wave is the rate at which the phase of the wave propagates in space, or in other words, it is the velocity of the wavefront propagation.

Consider position of the same phase, $\varphi = kx - \omega t + \varphi_0 = const$. If we differentiate this equation, we obtain $kdx - \omega dt = 0$, and the phase velocity

$$v_p = \frac{dx}{dt} = \frac{\omega}{k}. \tag{1.112}$$

A **plane wave** is a constant-frequency wave whose wavefronts (surfaces of constant phase) are infinite parallel planes of constant amplitude normal to the phase velocity vector.

A **spherical wave** is a constant-frequency wave whose wavefronts (surfaces of constant phase) are concentric spheres with the source of disturbance in the center. The phase velocity vector has radial direction from the source. Equation of the spherical wave is

$$u(\vec{r},t) = \frac{A}{r}\cos\left(\vec{k}\vec{r} - \omega t + \varphi_0\right). \tag{1.113}$$

Standing wave is the result of the superposition of two identical waves traveling in opposite directions.

Consider wave functions for two transverse sinusoidal waves having the same amplitude, frequency, and wavelength but traveling in opposite directions in the same medium $y_1 = A\sin\left(kx - \omega t\right)$ and $y_2 = A\sin\left(kx + \omega t\right)$. Adding these two functions gives the resultant wave function $y = y_1 + y_2 = A\sin\left(kx - \omega t\right) + A\sin\left(kx + \omega t\right)$. When we use the trigonometric formula, the expression reduces to the form

$$y = 2A\sin kx \cos \omega t, \tag{1.114}$$

which represents the wave function of a standing wave.

The points of zero amplitude of the standing wave are called **nodes**:

$$\sin kx = 0;\ x = \frac{\lambda}{2}\pi n,\ \ n = 0,1,2,...$$

1.7.2 Energy transfer by elastic wave

Waves transport energy through a medium as they propagate. As the external agent performs work on the end of the string, moving it up and down, energy enters the system of the string and propagates along its length. Let's focus our attention on an infinitesimal element of the string of length *dx* and mass *dm*. Each such element moves vertically with simple harmonic motion. Therefore, we can model each element of the string as a simple harmonic oscillator, with the oscillation in the *y* direction. All elements have the same angular frequency ω and the same amplitude *A*. The kinetic energy *dK* associated with the up and down motion of this element is

$dK = \dfrac{(dm)v_y^2}{2} = \dfrac{(\rho_l dx)v_y^2}{2}$, where ρ_l is the mass per unit length of the string.

Substituting for the general transverse speed of an element of the medium gives

$$dK = \frac{\rho_l}{2}\left(-\omega A\sin(kx - \omega t)\right)^2 dx = \frac{\rho_l \omega^2 A^2}{2}\sin^2(kx - \omega t)dx.$$

If we take a snapshot of the wave at time *t* =0, $dK = \dfrac{\rho_l \omega^2 A^2}{2}\sin^2(kx)dx$.

Integrating this expression over all the string elements in a wavelength of the wave gives the total kinetic energy K_λ in one wavelength:

$$K_\lambda = \frac{\rho_l \omega^2 A^2}{2}\int_0^\lambda \sin^2(kx)dx = \frac{\rho_l \omega^2 A^2 \lambda}{4}.$$

In addition to kinetic energy, there is potential energy associated with each element of the string due to its displacement from the equilibrium position and the restoring forces from neighboring elements. A similar analysis to that above for the total potential energy U_λ in one wavelength gives exactly the same result:

$$U_\lambda = \frac{\rho_l \omega^2 A^2 \lambda}{4}.$$

The total energy in one wavelength of the wave is the sum of the potential and kinetic energies:

$$E_\lambda = K_\lambda + U_\lambda = \frac{\rho_l \omega^2 A^2 \lambda}{2}. \tag{1.115}$$

Then, the power P, or **rate of energy transfer** associated with the mechanical wave, is

$$P = \frac{E_\lambda}{T} = \frac{\rho_l \omega^2 A^2}{2}\frac{\lambda}{T} = \frac{\rho_l \omega^2 A^2}{2}v. \tag{1.116}$$

1.7.3 Sound

Sound is the mechanical wave traveling through medium that typically results in the human perception of hearing.

As sound waves travel through air, elements of air are disturbed from their equilibrium positions leading to the changes in density and pressure of the air along the direction of wave motion.

(1) Audible waves lie within the range of sensitivity of the human ear. They have frequencies from about 16 Hz to 20000 Hz;

(2) Infrasonic waves have frequencies below the audible range;

(3) Ultrasonic waves have frequencies above the audible range.

Intensity I of the wave, or the power per unit area, is the rate at which the energy transported by the wave transfers through a unit area S perpendicular to the direction of travel of the wave.

$I = \frac{E}{St}$, where E is the energy transported through the area S in time t.

The intensity in the range the human ear can detect is characterized by the **sound level** β:

$\beta = 10 \cdot \lg \frac{I}{I_0}$, where I is the intensity of the sound;

$I_0 = 10^{-12}$ W/m^2 is the sound intensity at the threshold of hearing at frequency of 100 Hz.

The sound wave of a certain frequency is a **tone**.

Pitch of sound is an auditory sensation of how "low" or "high" a sound is according to its frequency (musical tone).

Echo is perception of a reflection of sound that arrives at the listener with a delay after the direct sound. The delay is proportional to the distance of the reflecting surface from the source and the listener.

1.7.4 Doppler effect

The **Doppler effect** (or the **Doppler shift**) is the change in frequency or wavelength of a wave for an observer moving relative to its source.

When the source of the waves is moving towards the observer, each successive wave crest is emitted from a position closer to the observer than the previous wave. Therefore, each wave takes slightly less time to reach the observer than the previous wave. Hence, the time between the arrival of successive wave crests at the observer is reduced, causing an increase in the frequency. While they are travelling, the distance between successive wave fronts is reduced, so the waves "bunch together". Conversely, if the source of waves is moving away from the observer, each wave is emitted from a position farther from the observer than the previous wave, so the arrival time between successive waves is increased, reducing the frequency. The distance between successive wave fronts is then increased, so the waves "spread out".

Suppose the source is in motion at speed v_s and the observer is at rest. Let's take the frequency of the source to be υ, the wavelength to be λ, and the speed of sound to be v. If the source moves directly toward observer, the wavelength λ' measured by observer is shorter than the wavelength λ of the source. During each vibration, which lasts for a time interval T (the period), the source moves a distance $v_sT = v_s/\upsilon$ and the wavelength is shortened by this amount. Therefore, the observed wavelength is $\lambda' = \lambda - \frac{v_s}{\upsilon}$. The frequency heard by observer is $\upsilon' = \frac{v}{\lambda'} = \frac{v}{\lambda - v_s/\upsilon} = \frac{v}{v/\upsilon - v_s/\upsilon}$. Or,

$$\upsilon' = \frac{v}{v - v_s}\upsilon \text{ (source moving toward observer)} \qquad (1.117)$$

When the source moves away from a stationary observer, the observer measures a wavelength λ' that is greater than λ and hears a decreased frequency:

$$\upsilon' = \frac{v}{v + v_s}\upsilon \text{ (source moving away from observer)} \qquad (1.118)$$

Now let's suppose the observer is in motion and the source is at rest. Let's take the frequency of the source to be υ, the wavelength to be λ, and the speed of sound to be v. When the observer moves toward the source, the speed of the waves relative to the observer is $v' = v + v_0$, where v_0 is the speed of the observer, but the wavelength λ is unchanged. Hence, we can say that the frequency heard by the observer is increased and is given by $\upsilon' = \frac{v'}{\lambda} = \frac{v + v_0}{\lambda}$. Or,

$$\upsilon' = \frac{v + v_0}{v}\upsilon \text{ (observer moving toward source).} \qquad (1.119)$$

If the observer is moving away from the source, the speed of the wave relative to the observer is $v' = v - v_0$. The frequency heard by the observer in this case is decreased and is given by

$$\upsilon' = \frac{v - v_0}{v}\upsilon \text{ (observer moving away from source).} \quad (1.120)$$

Questions for self-control

1. What is the elastic wave?
2. Distinctions between the longitudinal and transversal waves
3. Equation of the plane and spherical waves.
4. Standing waves.
5. Phase velocity of the waves.
6. Energy of the elastic wave.
7. What is the sound?
8. . Doppler effect.

Topic 1.8. Fundamentals of special relativity

Lecture 9

1.8.1 Fundamentals of the special relativity theory

Special relativity is the generally accepted and experimentally confirmed physical theory regarding the relationship between space and time, which was originally proposed in 1905 by Albert Einstein.

Two postulates of special relativity:

1. The principle of relativity: The laws of physics must be the same in all inertial reference frames. That is, all the laws of physics—those dealing with mechanics, electricity and magnetism, optics, thermodynamics, and so on—are the same in all reference frames moving with constant velocity relative to one another. This postulate is a generalization of the principle of Galilean relativity, which refers only to the laws of mechanics.

2. The constancy of the speed of light: The speed of light in vacuum has the same value, $c = 3.00 \cdot 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

1.8.2 Lorentz's transformations

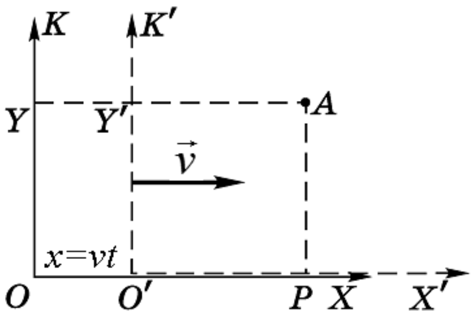


Figure 1.46.

An important consequence of the special relativity theory is the **Lorentz transformation** of coordinates and time between two coordinate frames that move at constant velocity relative to each other.

Consider two inertial frames of reference K and K' (Figure 1.46), where the frame K' is moving along the axis x at constant speed v with respect to the frame K. Galilean transformation of Newtonian physics assume an absolute space and time for these frames $t = t'$. But, the Galilean transformation is not valid when v approaches the speed of light, and the Lorentz transformation must be introduced. In the Lorentz case, the value for t' assigned to an event in the frame K' depends both on the time t and on the coordinate x as measured in the frame K. In other words, in relativity, space and time are not separate concepts but rather are closely interwoven with each other.

Let us consider these transformations. Let's assume that in the moment of time $t = t' = 0$ the coordinates of the system K and K' coincide (Figure 1.46). Suppose that the system K' moves to the right along the axis x at constant velocity v. As the relative velocity is directed along the axis x, other coordinates $y = y'$; $z = z'$. Let us see how the coordinate x changes with time. Point O' is the origin of the system K', that is $x' = 0$. Coordinate of this point relative to the fixed system K at the moment of time t is $x = vt$. Then, $x - vt = 0$.

We can assume that x' and $x - vt$ for any moment of time differ only by a constant factor γ, as far as space is homogeneous:

$$x' = \gamma\left(x - vt\right). \tag{1.121}$$

Now consider the point O, which corresponds to the origin of the fixed system K: its coordinate in this system is $x = 0$. In a movable system the same point in the moment of time t' has the coordinate $x' = -vt'$. Then, $x' + vt' = 0$. We can assume as before that $x = \gamma\left(x' + vt'\right)$.

Considering the equivalence of the two inertial systems it is possible to show that the proportionality coefficients γ have to be same. Let's find the factor γ, considering the fact that the velocity of light in both systems has the same magnitude c.

Suppose that at the time $t = t' = 0$ from the points O and O', which coincide, a light signal is emitted. During the time t signal will reach a certain point with coordinate $x = ct$. In the K' system the coordinate of this point is $x' = ct'$. Then,

$$x' = ct' = \gamma(x - vt) = \gamma(ct - vt) = \gamma(c - v)t;$$

$$x = ct = \gamma(x' + vt') = \gamma(ct' + vt') = \gamma(c + v)t'.$$

Let's take a product of these equations, $c^2 t' t = \gamma^2 (c + v)(c - v) t' t$, from which,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{1.122}$$

Now we obtain formula for the **coordinate transformation**:

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}; \qquad x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \tag{1.123}$$

Now let's find expressions for the transformation of time t and t'.

$$x = \gamma(x' + vt') = \gamma(\gamma(x - vt) + vt') = \gamma^2 x - \gamma^2 vt - \gamma vt';$$

$$\gamma vt' = x + \gamma^2 vt - \gamma^2 x = x(1 - \gamma^2) + \gamma^2 vt;$$

$$t' = \gamma t + \frac{1 - \gamma^2}{\gamma v} x = \gamma\left(t + \frac{1 - \gamma^2}{\gamma^2 v} x\right) = \gamma\left(t + \left(\frac{1}{\gamma^2} - 1\right)\frac{x}{v}\right) = \gamma\left(t - \frac{v}{c^2} x\right);$$

By substituting γ, we obtain **time transformations**:

$$t' = \frac{t - \frac{v}{c^2} x}{\sqrt{1 - v^2/c^2}}; \qquad t = \frac{t' + \frac{v}{c^2} x'}{\sqrt{1 - v^2/c^2}}. \tag{1.124}$$

Thus, we have proved that transformation formulas for the transition from the K' system into the K system look like

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}};\ y = y';\ z = z';\ t = \frac{t' + \frac{v}{c^2} x'}{\sqrt{1 - v^2/c^2}}. \tag{1.125}$$

Reverse transformation from K system into K' system:

$$x'=\frac{x-vt}{\sqrt{1-v^2/c^2}};\ y=y';\ z=z';\ t'=\frac{t-v/c^2\,x}{\sqrt{1-v^2/c^2}}. \qquad (1.126)$$

These equations are called the **Lorentz transformation**.

It can be easily understood that when $v \ll c$, Lorentz's transformations transform into Galilean transformations. That is, Galilean transformations are valid only in case of motion with speed much smaller than the speed of light. One more conclusion is that motion with speed exceeding the speed of light in a vacuum $v > c$ is impossible.

1.8.3 Consequences of Lorentz's transformations. The relativity of simultaneous events in different frames of reference

Consider how events depend on time and space in different inertial reference systems. Newtonian mechanics states that a universal time scale exists that is the same for all observers. But, when the motion with the speed close to the speed of light takes place, the two events that appear to be simultaneous to the observer at rest do not appear to be simultaneous to the observer in motion. The two observers must find that light travels at the same speed, so observation of the event depends on the transit time of light from the event to the observers.

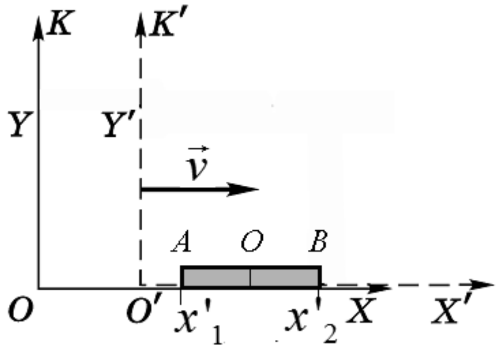


Figure 1.47.

Special relativity theory leads to the conclusion that time is not absolute and its measure depends on the frame of reference K and K'. Frame K' moves relative to K along the x-axis to the right with velocity v as shown in the Figure 1.47. Let a rod in system K' be at rest.

There is a flashlight at point *O* on the rod, and there are photocells at the ends at points *A* and *B*. Coordinate of the point *A* in the system K' is x'_1, and coordinate of the point *B* is x'_2. From the point of view of the observer, who is in the movable system K', the light flash reaches photocells *A* and *B*, which are equidistant from the point *O*, at one and the same moment of time $t'_1 = t'_2 = t'$ (or $\Delta t' = t'_2 - t'_1 = 0$), as the speed of light transmission is the same in both directions and equals c. Thus, in the frame K' both photocells will operate simultaneously at time $t'_1 = t'_2 = t'$. Now consider this situation from the point of view of the observer in the frame K. The speed of light must also be the same in all directions, but distances covered by light to the photocells *A* and *B* are different now, photocell *A* is moving (relative to frame K) toward the light signal and the photocell *B* is moving away from the light signal. Therefore, in the frame K the signal will reach the point *A* faster than the point *B*.

Assume that the coordinate x'_1 in the frame K' corresponds to the *x* coordinate in the frame K, similarly, the coordinate x_2 corresponds to the coordinate x'_2. Relative to the observer in the frame K a photocell *A* is triggered at the moment

$$t_1 = \frac{t'_1 + v\, x'_1 / c^2}{\sqrt{1 - v^2/c^2}} = \frac{t' + v\, x'_1 / c^2}{\sqrt{1 - v^2/c^2}}.$$

And photocell *B* at the moment

$$t_2 = \frac{t'_2 + v\, x'_2 / c^2}{\sqrt{1 - v^2/c^2}} = \frac{t' + v\, x'_2 / c^2}{\sqrt{1 - v^2/c^2}}.$$

Then, the **<u>relation between time and space</u>**:

$$\Delta t = t_2 - t_1 = \frac{v/c^2 \left(x'_2 - x'_1\right)}{\sqrt{1 - v^2/c^2}}. \qquad (1.127)$$

When the simultaneous events in the K' frame are separated in space $x'_1 \neq x'_2$ they will be not simultaneous in the K' frame ($t_1 \neq t_2$). If $t_2 - t_1$ becomes negative, it indicates a change in the order of events, that is in some frames event *1* would be earlier than event *2*, and in others, by contrast, event *2* would be earlier than event *1*. But the cause-connected events (e.g. shot and hitting the ball) in any frame of reference can not exchange places and always an event that is the reason is earlier than the effect.

Thus, for high speed $v \to c$ the events simultaneous in one reference frame may be not simultaneous in other frames, that is simultaneity is a relative concept.

1.8.4 The Effect of time relativity

The fact that the events that occur simultaneously in one reference frame, may not be simultaneous in another frame, makes it possible to believe that time itself is not absolute and flows depending on the choice of reference frame. Observers in different inertial frames can measure different time intervals between a pair of events.

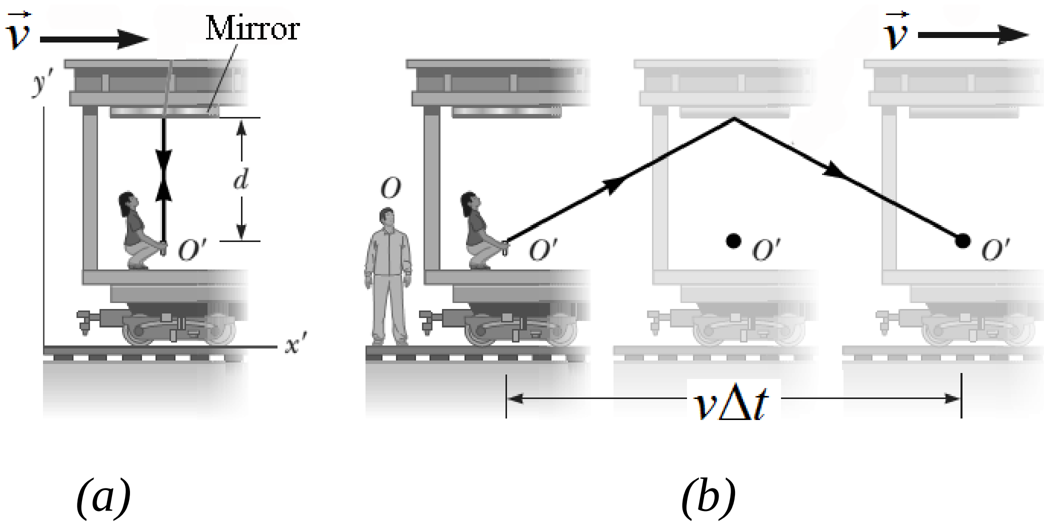


Figure 1.48.

Consider a vehicle moving to the right with a speed *v* (Figure 1.48). A mirror is fixed to the ceiling of the vehicle, and observer *O'* at rest in the frame attached to the vehicle holds a flashlight a distance *d* below the mirror. At some instant, the flashlight emits a pulse of light directed toward the mirror (event 1), and at some later time after reflecting from the mirror, the pulse arrives back at the flashlight (event 2). Observer *O'* carries a clock and uses it to measure the time interval Δt_0 between these two events. Because the light pulse has a speed *c*, the time interval required for the pulse to travel from *O'* to the mirror and back is $\Delta t_0 = \dfrac{2d}{c}$.

Now consider the same pair of events as viewed by observer *O* in a second frame at rest with respect to the ground. According to this observer, the mirror and the flashlight are moving to the right with a speed *v*, and as a result, the sequence of events appears entirely different. By the time the light from the flashlight reaches the mirror, the mirror has moved to the right a distance $v\Delta t/2$, where Δt is the time interval required for the light to travel from *O'* to the mirror and back to *O'* as measured by *O*. Observer *O* concludes that because of the motion of the vehicle, if the light is to hit the mirror, it must leave the flashlight at an angle with respect to the vertical direction. The light must travel farther in part (b) than in part (a) (Figure 1.48). According to the second postulate of the special theory of relativity, both observers must measure *c* for the speed of light. Because the light travels farther according to *O*, the time interval Δt measured by *O* is longer than the time interval Δt_0 measured by *O'*. To obtain a relationship between these two time intervals, let's use the Pythagorean theorem:

$$d^2 + \left(\frac{v\Delta t}{2}\right)^2 = \left(\frac{c\Delta t}{2}\right)^2.$$

From which

$$\Delta t = \frac{2d}{c}\frac{1}{\sqrt{1 - v^2/c^2}}.$$

Knowing that $\frac{2d}{c} = \Delta t_0$, we obtain the **time dilation**:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} = \gamma \Delta t_0. \tag{1.128}$$

The magnitude $\sqrt{1 - v^2/c^2} < 1$, that's why $\Delta t >> \Delta t_0$. Time interval Δt measured by an observer moving with respect to a clock is longer than the time interval Δt_0 (called a **proper time**), measured by the observer at rest with respect to the clock. This effect is known as **time dilation**.

Thus, time intervals depend on the velocity *v* of the frame of reference with respect to other frames. If a clock is moving with respect to you, the time interval between ticks of the moving clock is observed to be longer than the time interval between ticks of an identical clock in your reference frame. Therefore, it is often said that a moving clock is measured to run more slowly than a clock in your reference frame by a factor γ. We can generalize this result by stating that all physical processes, including mechanical, chemical, and biological ones, are measured to slow down when those processes occur in a frame moving with respect to the observer.

This relationship is essential for the velocities v, close to the velocity of light c. In classical mechanics $v \ll c$ and $\Delta t = \Delta t_0$, that is a time intervals do not depend on the state of motion.

1.8.5 Length Contraction

The measured distance between two points in space also depends on the frame of reference of the observer. The **proper length** l_0 of an object is the length measured by an observer at rest relative to the object. The length of an object measured by someone in a reference frame that is moving with respect to the object is always less than the proper length. This effect is known as **length contraction**.

To understand length contraction, consider a frame of reference K', which moves relative to the frame K with a constant velocity v, and a rod at rest in the frame K'. Let the rod be oriented along the axis x'. The coordinates of the rod ends are x'_1 and x'_2 in this reference frame. The proper length of the rod is $l_0 = x'_2 - x'_1$ in the frame K'. Let's determine the length of the same rod, if it is measured in the reference frame K. For the observer in the reference frame K the rod will move with velocity v. To measure the length of the rod, which is moving, the observer must mark on the x-axis the position of its ends x_2 and x_1 in the frame K at the same time. Their difference $l = x_2 - x_1$ determines the length of the rod in the frame K. But the coordinates x_1 and x_2 are connected with the coordinates x'_1 and x'_2 by the Lorenz's relations $x'_2 = \dfrac{x_2 - vt}{\sqrt{1 - v^2/c^2}}$; $x'_1 = \dfrac{x_1 - vt}{\sqrt{1 - v^2/c^2}}$.

Thus, the proper length of the rod is

$$l_0 = x'_2 - x'_1 = \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}} - \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} = \frac{l}{\sqrt{1 - v^2/c^2}}.$$

The length of the rod in the frame K is

$$l = l_0 \cdot \sqrt{1 - v^2/c^2}\ . \tag{1.129}$$

Thus, linear dimension of a body in the direction of its motion is shrinking relative to the reference frame K $\sqrt{1-v^2/c^2}$ times and is less than the proper length of the body l_0 ($l << l_0$). The length of the rod is the longest in the frames of reference with respect to which the body is at rest. This result is called the Lorentz's contraction.

From the Lorentz' transformations it follows that $y'_2 - y'_1 = y_2 - y_1$ and $z'_2 - z'_1 = z_2 - z_1$, that is the transverse sizes of the body do not depend on the velocity of its motion and are equal in all inertial frames of reference.

1.8.6 The interval between two events

Consider the events *1* and *2* that are taking place in the inertial frame of reference K at a point $A(x_1, y_1, z_1)$ at time t_1 and at a point $B(x_2, y_2, z_2)$ at time t_2 respectively. In the frame of reference K' these events are taking place at the point $A(x'_1, y'_1, z'_1)$ at time t'_1 and at the point $B(x'_2, y'_2, z'_2)$ at time t'_2. Denote: $l_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ – the squared distance between points *A* and *B* in the frame K; $l'_{12} = (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2$ – the squared distance between points A' and B' in the frame K'.

With the help of Lorentz transformation it is easy to prove that

$$l_{12}^2 - c^2 t_{12}^2 = l'^2_{12} - c^2 l'^2_{12},$$

where $t_{12} = t_2 - t_1$, $t'_{12} = t'_2 - t'_1$ – time intervals between the events in both frames of reference.

Magnitude

$$S_{12} = \sqrt{c^2 t_{12}^2 - l_{12}^2} \tag{1.130}$$

is called an **interval between two events**. The interval between two events is the same in any inertial reference frames: $S_{12} = S'_{12}$. The interval $S_{12} = \sqrt{c^2 t_{12}^2 - l_{12}^2}$ can be real $S_{12}^2 > 0$, imaginary $S_{12}^2 < 0$, and equal to zero.

If $S_{12}^2 < 0$, then $|l_{12}| > c|t_{12}|$. This imaginary interval is called space-like. In this case, the events under consideration will not affect each other because signals cannot be extended at a velocity that exceeds *c*. For the events, separated by the space-like intervals, there is a frame of reference where they can take place simultaneously: if $t'_{12} = 0$, then an interval $c^2 t_{12}^2 - l_{12} = -l'_{12}$, if $S_{12}^2 > 0$, then $|l_{12}| \le c|t_{12}|$. Such a real interval is called a time-like. The events, separated by this interval, can be etiological. For such events there are no frames of reference in which they occur simultaneously: if $t'_{12} = 0$, then the interval becomes real. It contradicts the definition of time-like interval that must be real. The development of this statement makes it possible to show that the reason can never be earlier than the effect.

1.8.7 The velocity conversion. Relativistic rule of velocity addition

The conversion of velocity in the case of transition from one inertial frame to another in classical mechanics is realized according to the rule

$$\vec{v} = \vec{v}' + \vec{v}_0;\ v_x = v_x' + v_0;\ v_y = v'_y;\ v_z = v'_z,$$

in the case when the frame K' moves along the *x*-axis at constant velocity v_0.

Let us assume that in the K' frame the speed of light $v' = c$, then, according to the rule of velocity addition in classical mechanics, $v_x = c + v_0$, i.e. the speed of light is exceeded. This is impossible, so let's determine new rules of velocity transformation which would satisfy the second postulate of the relativity theory.

Let's use Lorentz's transformation laws:

$$x = \frac{x' + v_0 t'}{\sqrt{1 - v_0^2 / c^2}};\ \ y = y';\ \ z = z';\ \ t = \frac{t' + \frac{v_0}{c^2} x'}{\sqrt{1 - v_0^2 / c^2}}.$$

Let's differentiate the equations:

$$dx = \frac{dx' + v_0 dt'}{\sqrt{1 - {v_0}^2 / c^2}}; \quad dy = dy'; \quad dz = dz'; \quad dt = \frac{dt' + {v_0}/{c^2}\, dx'}{\sqrt{1 - {v_0}^2 / c^2}}.$$

Having divided the expressions by dx, dy and dz by dt we'll get

$$\frac{dx}{dt} = \frac{dx' + v_0 dt'}{dt' + \frac{v_0}{c^2} dx'} = \frac{\frac{dx'}{dt'} + v_0}{1 + \frac{v_0}{c^2}\frac{dx'}{dt'}}; \quad \frac{dy}{dt} = \frac{dy' \sqrt{1 - \frac{v_0^2}{c^2}}}{dt' + \frac{v_0}{c^2} dx'} = \frac{\frac{dy'}{dt'}\sqrt{1 - \frac{v_0^2}{c^2}}}{1 + \frac{v_0}{c^2}\frac{dx'}{dt'}};$$

$$\frac{dz}{dt} = \frac{dz' \sqrt{1 - \frac{v_0^2}{c^2}}}{dt' + \frac{v_0}{c^2} dx'} = \frac{\frac{dz'}{dt'}\sqrt{1 - \frac{v_0^2}{c^2}}}{1 + \frac{v_0}{c^2}\frac{dx'}{dt'}}.$$

Taking into account that $\frac{dx}{dt} = v_x$; $\frac{dx'}{dt'} = v'_x$; $\frac{dy}{dt} = v_y$; $\frac{dy'}{dt'} = v'_y$; $\frac{dz}{dt} = v_z$; $\frac{dz'}{dt'} = v'_z$, we obtain the formulae for velocity transformation in the Einstein theory:

$$v_x = \frac{v'_x + v_0}{1 + \frac{v_0}{c^2} v'_x}; \quad v_y = \frac{v'_y \sqrt{1 - \frac{v_0^2}{c^2}}}{1 - \frac{v_0}{c^2} v'_x}; \quad v_z = \frac{v'_z \sqrt{1 - \frac{v_0^2}{c^2}}}{1 - \frac{v_0}{c^2} v'_x}. \qquad (1.131)$$

Similarly, for the transfer from the K frame to the K' frame

$$v'_x = \frac{v_x - v_0}{1 - \frac{v_0 v_x}{c^2}}; \quad v'_y = \frac{v_y \sqrt{1 - \frac{v_0^2}{c^2}}}{1 - \frac{v_0 v_x}{c^2}}; \quad v'_z = \frac{v_z \sqrt{1 - \frac{v_0^2}{c^2}}}{1 - \frac{v_0 v_x}{c^2}}. \qquad (1.132)$$

At small velocities $v << c$ (if $\sqrt{1 - \frac{v_0^2}{c^2}} \to 1$) we get the expressions for the velocity addition in classical mechanics. Supposing that in the K' frame along the x

axis light travels at the speed $v'_x = c$, let's find the speed of light in the K frame. We get $v_x = \dfrac{c + v_0}{1 + \dfrac{v_0 c}{c^2}} = c$. The speed of light in the K frame equals c, as well as in the K' frame.

Questions for self-control

1. Einstein's postulates of the special relativity.
2. Difference between the Galilean and the Lorentz transformations for time and coordinates.
3. Time dilation
4. Length Contraction
5. Simultaneousness of events in different inertial reference frames.
6. Relation between time and space.
7. Relativistic rule of velocity addition.
8. Time interval between two events.

Lecture 10

1.8.8 Relativistic dynamics. Newton's second law of motion and the momentum in relativistic mechanics

To describe the motion of particles within the framework of the special theory of relativity properly, you must replace the Galilean transformation equations by the Lorentz transformation equations. Because the laws of physics must remain unchanged under the Lorentz transformation, we must generalize Newton's laws and the definitions of linear momentum and energy to conform to the Lorentz transformation equations and the principle of relativity.

First, recall from the isolated system model that when two particles collide, the total momentum of the isolated system of the two particles remains constant. Suppose we observe this collision in a reference frame *K* and confirm that the momentum of the system is conserved. Now imagine that the momenta of the particles are measured by an observer in a second reference frame *K'* moving with velocity $\vec{v}_0$ relative to the first frame. Using the Lorentz velocity transformation equation and the classical definition of linear momentum, we find that linear momentum is not measured to be conserved by the observer in *K'*. Because the laws of physics are the same in all inertial frames, however, linear momentum of the system must be conserved in all frames. To solve this contradiction we must modify the definition of linear momentum so that the momentum of an isolated system is conserved for all observers. For any particle, the correct **relativistic equation for linear momentum** that satisfies this condition is

$$\vec{p} = \gamma m\vec{v} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{1.133}$$

where *m* is the mass of the particle and $\vec{v}$ is the velocity of the particle.

Let’s derive this expression. Suppose that in some inertial reference frame K two identical particles *1* and *2* are moving opposite to each other with speed v. Consider the instance when they collide in the absolutely elastic collision.

If the velocity component of the first particle along the y axis before the collision was $-v_y$, after the collision it becomes equal to v_y. The projections of the first and second particles momenta before the collision are $-mv_y$ and mv_y, respectively. The total momentum of the system of two particles in the direction of y before the collision is mv_y-mv_y=0. After the collision, the y projections of the first and second particles momenta are mv_y, and $-mv_y$, respectively. The total momentum after the collision in the direction of y is equal to 0. Likewise, it can be shown that for the x axis the momentum of the system does not change. Therefore, in the K frame the law of conservation the momentum is valid.

Let’s consider the instance of collision from the point of view of the observer, who is at rest in the K' frame, which moves at the velocity $v = v_x$ relating to the K frame. Here v_x is the x projection of the second particle’s velocity before the collision. As a result of such choice of the K' frame, the second particle has only y velocity component. If the velocity in the K frame is equal to v_y, then from the equation of relativistic velocity addition it follows that

$$v'_y = \frac{v_y\sqrt{1-\frac{v_0^2}{c^2}}}{1-v\frac{v_x}{c^2}}.$$

Let’s write y as a velocity component of the particles before the collision in the K' frame. Taking into account that $v = v_x$:

$$-v'^{(1)}_y = \frac{-v_y\sqrt{1-\frac{v_x^2}{c^2}}}{1+\frac{v_x^2}{c^2}}; \qquad v'^{(2)}_y = \frac{v_y\sqrt{1-\frac{v_x^2}{c^2}}}{1-\frac{v_x^2}{c^2}}.$$

After the collision,

$$v''^{(1)}_y = \frac{v'_y \sqrt{1-\frac{v_x^2}{c^2}}}{1+\frac{v_x^2}{c^2}}; \qquad v''^{(2)}_y = -\frac{v'_y \sqrt{1-\frac{v_x^2}{c^2}}}{1-\frac{v_x^2}{c^2}}.$$

We see that the total linear momentum in the y direction before and after the collision is not equal. At the same time the linear momentum along the x-axis is conserved: $v'^{(1)}_x = v''^{(1)}_x$. It means that we must find the expression for the linear momentum which will be Lorentz-invariant.

According to Lorentz transformation laws the displacements Δy along the y-axis are equal in different reference frames. But time Δt necessary for passing the distance Δy depends on the choice of the reference frame. Thus, the velocity component $v_y = \frac{\Delta y}{\Delta t}$ will be different for different reference frames.

For measuring the time interval we use the imaginary watch located on the particle. The watch will measure the proper time of the particle Δt_0:

$$\Delta t_0 = \Delta t \sqrt{1 - v^2/c^2},$$

from which

$$\frac{\Delta y}{\Delta t_0} = \frac{\Delta y}{\Delta t}\frac{\Delta t}{\Delta t_0} = \frac{\Delta y}{\Delta t}\sqrt{1-v^2/c^2} = \frac{v_y}{\sqrt{1-v^2/c^2}}.$$

Since $\frac{\Delta y}{\Delta t_0}$ is equal for different reference frames, the y component of the vector $\frac{\vec{v}}{\sqrt{1-v^2/c^2}}$ is equal for all the reference frames which differ only by their velocity components along the x-axis. Having multiplied the vector quantity $\frac{\vec{v}}{\sqrt{1-v^2/c^2}}$ by the mass of a particle m_0, we get the relativistic momentum

$\vec{p}=\frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}$, where m_0 is the mass of a particle at rest.

With such definition of the linear momentum, both y and x projections of the vector $\vec{p}$ remain unchanged in any inertial reference frame, thereby, the law of conservation the linear momentum in relativistic mechanics is valid. At small velocities $\frac{v}{c}<<1$ we get the expression for the momentum in classical mechanics: $\vec{p}=m_0\vec{v}$.

Newton's second law of motion $\vec{F}=\frac{d\vec{p}}{dt}$ turns to be Lorentz-invariant if we consider the momentum of the body as $\vec{p}=\frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}$.

We obtain the **basic equation of relativistic dynamics**:

$$\vec{F}=\frac{d\vec{p}}{dt}=\frac{d}{dt}\left(\frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}\right) \tag{1.134}$$

1.8.9 Relation between the energy and mass in special theory of relativity

Let's find the expression for the kinetic energy of a particle in relativistic mechanics. Change in the kinetic energy of the particle dE_K in the case of displacement for the distance $d\vec{r}$ is equal to work $\vec{F}d\vec{r}$ done by force $\vec{F}$ on the particle $dE_K=\delta A=\vec{F}d\vec{r}$, or $dE_K=\left(\vec{F}\vec{v}\right)dt$, since $d\vec{r}=\vec{v}dt$, where $\vec{v}$ is the velocity of the particle.

As far as $\vec{F}dt = d\vec{p}$, we'll get

$$dE_K = \vec{v}d\left(\frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}\right) = \frac{\vec{v}\left(m_0 d\vec{v}\sqrt{1-v^2/c^2} + m_0\vec{v}d\vec{v}\left(\frac{vdv}{c^2}\right)\left(1-v^2/c^2\right)^{-\frac{1}{2}}\right)}{1-v^2/c^2} =$$

$$= \vec{v}\left(\frac{m_0 d\vec{v}}{\sqrt{1-v^2/c^2}} + \frac{m_0\vec{v}\left(vdv/c^2\right)}{\left(1-v^2/c^2\right)^{\frac{3}{2}}}\right).$$

Taking into account that $\vec{v}d\vec{v} = vdv$ we get

$$dE_K = \frac{m_0 d\left(v^2/2\right)}{\sqrt{1-v^2/c^2}} + \frac{m_0 v^2 d\left(v^2/2c^2\right)}{\left(1-v^2/c^2\right)^{\frac{3}{2}}} =$$

$$\frac{m_0 d\left(v^2/2\right)\left(1-v^2/c^2 + v^2/c^2\right)}{\left(1-v^2/c^2\right)^{\frac{3}{2}}} = \frac{m_0 d\left(v^2/2\right)}{\left(1-v^2/c^2\right)^{\frac{3}{2}}} = \frac{c^2 \cdot m_0 d\left(v^2/c^2\right)}{\left(1-v^2/c^2\right)^{\frac{3}{2}}} = d\left(\frac{m_0 c^2}{\sqrt{1-v^2/c^2}}\right).$$

After performing integration, we obtain

$E_K = \dfrac{m_0 c^2}{\sqrt{1-v^2/c^2}} + const$, where $const$ is the constant of integration.

The constant of integration can be found from the condition that if the velocity $v = 0$, then the kinetic energy of the particle is also zero: $E_K = 0$. Then, $const = -m_0 c^2$.

The constant of integration $m_0 c^2$ has dimension of energy and is called the **rest energy** of the body:

$$E_0 = m_0 c^2. \tag{1.135}$$

The rest energy allows considering any physical body as some kind of reservoir for energy, which can be easily transformed into any other kind of energy. Thus, the expression for the relativistic kinetic energy takes form

$$E_K = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2. \tag{1.136}$$

At small velocities under the condition that $v << c$

$$E_K = m_0 c^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) \approx m_0 c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1 \right) = \frac{1}{2} m_0 v^2.$$

Let’s denote $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$.

This value of the mass m is called the **relativistic mass** of the body. We can write

$$E_K = mc^2 - m_0 c^2,$$

or

$$mc^2 = E_K + m_0 c^2 = E. \tag{1.137}$$

The sum of kinetic energy and rest energy is called the total energy E of the body. The total energy of the body, as well as the rest energy, does not include the potential energy of the external force field.

The decrease of the total energy of the body (system) leads to the equivalent change of its mass $\Delta m = \dfrac{\Delta E}{c^2}$ In the course of usual macroscopic processes the mass of a body slightly changes and it cannot be measured.

The mass-energy equivalence principle was first experimentally discovered and proved in nuclear physics. It was due to the fact that, nuclear processes and the processes of elementary particles transformation are accompanied with sufficiently large energy changes, which can be compared to the rest energy of the particles.

1.8.10 The energy-momentum relation

Let's find the energy-momentum relation in relativistic dynamics. It is known that energy $E = mc^2$ and momentum $\vec{p} = m\vec{v}$, where $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$; $\vec{v}$ is the velocity of the body.

Let's square the equations of energy and momentum and multiply them by c^2. We obtain

$$E^2 = m^2c^4;\ p^2c^2 = m^2v^2c^2.$$

$$E^2 - p^2c^2 = m^2c^4\left(1 - \frac{v^2}{c^2}\right) = \left(m_0 \Big/ \sqrt{1 - \frac{v^2}{c^2}}\right)^2 c^4\left(1 - \frac{v^2}{c^2}\right) = m_0^2 \cdot c^4.$$

In such a way we get the energy-momentum relation:

$$E^2 - p^2c^2 = m_0^2c^4.$$

The rest mass m_0 and the speed of light in vacuum c are invariable for all the inertial frames. These magnitudes are also Lorentz-invariant. Thus, energy E and momentum p can change in case of passing from one inertial frame to another, but the difference $E^2 - p^2c^2$ remains invariable for all reference frames. This significant peculiarity of the energy-momentum relation can be expressed:

$$E^2 - p^2c^2 = const = inv.$$

The **relativistic energy**:

$$E = \sqrt{p^2c^2 + m_0^2c^4}. \qquad (1.138)$$

Questions for self-control

1. Relativistic mass.
2. Relativistic linear momentum
3. Basic equation of relativistic dynamics
4. Energy of relativistic particle.
5. Rest energy
6. Relationship between mass and energy.
7. The energy - momentum relation.

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